SUBDIFFUSION MODEL WITH TIME-DEPENDENT DIFFUSION COEFFICIENT
Integral-Balance Solution and Analysis

by

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The paper addresses approximate integral-balance approach to a time-fractional diffusion equation of order $0 < \mu < 1$ with a time-dependent diffusion coefficient of power-law type $D(t) = D_0 t^\beta$ where $0 < \beta < 1$. The form of the solution spreading in a semi-infinite medium through an analysis of the second moment of the approximate solution reveals that depending on the sum $\mu + \beta$ the solution can model subdiffusive ($\mu + \beta < 1$), superdiffusive ($\mu + \beta > 1$), or Gaussian ($\mu + \beta = 1$) process of transport. The optimal exponents of the approximate parabolic profiles have been determined by minimization the mean squared error of approximation over the penetration depth.

Key words: time-fractional diffusion, integral-balance method, time-dependent diffusivity, approximate solutions

Introduction

Fractional differential equations (FDE) are suitable for modelling of anomalous diffusive processes such as subdiffusion and superdiffusion in non-homogeneous porous media [1], turbulent transports [2], in both classical no-local [3] and local [4] sense. Analytical solutions for FDE are difficult to obtain [5] and most of them are approximate in nature [6-10] resulting in forms unsuitable for post-solution applications and engineering analysis. The common approaches are numerical methods such as: Galerking method [11], Tau method [12], and spectral-method [13] while analytical solutions are rare [14-16].

Most of FDE are with constant [17, 18], concentration-dependent [19] or space-dependent diffusion coefficients [15, 16] depending on the specific features of the transport process modelled.

This communication addresses integral-balance approach to 1-D linear subdiffusion equation with a time-dependent diffusion coefficient of power law type $D(t) = D_0 t^\beta$ [m² s⁻(1−β)]:

\[
\frac{\partial^\mu u(x,t)}{\partial t^\mu} = D(t) \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = 0, \quad u(0,t) = 1, \quad u(\infty,t) = 0 \quad (1a)
\]

\[
D(t) = D_0 t^\beta, \quad \text{with} \quad \mu \geq 0 \quad \text{and} \quad x > 0 \quad (1b)
\]

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The time-fractional derivative in eq. (1a) is left-sided of either of Riemann-Liouville (RL) eq. (2a) or Caputo type eq. (2b) [20]:

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} = D_t^\mu \left[ \frac{1}{\Gamma(1-\mu)} \int_0^t f(x,t) \, \frac{d^z f(x,z)}{dz^{\mu}} \, dz \right], \quad 0 < \mu < 1$$ (2a)

$$\frac{\partial^\mu u(x,t)}{\partial t^\mu} = c D_t^\mu \left[ \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{d}{dz} f(x,z) \, dz \right], \quad 0 < \mu < 1$$ (2b)

The problem (1a) and (1b) was conceived by Fa and Lenzi [21] and solved by a series expansion technique. The efforts of this analysis were oriented to the first passage time (FPT) in finite and infinite domains. Now, we address an approximate analytical solution based on the integral-balance approach [22, 23] successfully applied previously to solve models with time-fractional [24, 25], and space-fractional derivatives [26, 27]. The solution the Dirichlet problem of eq. (1) with zero initial conditions and RL time-fractional derivative is demonstrating the feasibility of the method.

Solution of the Dirichlet problem

The integral-balance approach to the subdiffusion equation

Consider eq. (1a) with initial and boundary conditions (Dirichlet problem):

$$u(x,0) = 0, \quad u(0,t) = 1, \quad \text{and} \quad u(\infty,t) = 0 \quad \text{for} \quad t > 0$$ (4a,b,c)

The integral-balance method uses the concept of a finite penetration depth $\delta(t)$ thus defining a sharp front of propagation of the solution requiring the boundary condition $u(\infty,t) = 0$ to be replaced by:

$$u(0) = 1, \quad u(\delta) = \frac{\delta}{\delta x}, \quad \text{and} \quad u(\delta) = 0$$ (5a,b,c)

The single integration approach known as the heat-balance integral method (HBIM) [22, 23] suggests integration over the entire penetration depth with respect to the space co-ordinate, $x$, namely:

$$\int_0^\delta \frac{\partial^\mu u(x,t)}{\partial t^\mu} \, dx = \int_0^\delta D(t) \frac{\partial^2 u(x,t)}{\partial x^2} \, dx \Rightarrow \int_0^\delta \frac{\partial^\mu u(x,t)}{\partial t^\mu} \, dx = -D(t) \frac{\partial u(0,t)}{\partial x}$$ (6a,b)

Further, the integral-balance method suggests replacement of the function $u(x,t)$ by an assumed profile $u_0(x,t) = u_0(x/\delta)$ as a function of the dimensionless pace variable $0 < x/\delta < 1$. This approach leads to an equation about the time evolution of the penetration depth, $\delta(t)$.

However, with HBIM the gradient at $x = 0$ in the right-hand side of eq. (6b) should be determined through the assumed approximate profile. This drawback can be avoided by the double-integration method (DIM) conceived for non-linear problems of integer-order ($\mu = 1$) [28, 29] and successfully applied to time-fractional equations [24, 25].

In accordance with DIM, the first step is an integration of eq. (1a) from 0 to $x$, namely:
Equation (6b) of HBIM can be expressed equivalently:

\[
\int_0^\delta \frac{\partial^n u(x,t)}{\partial t^n} \, dx + \int_0^\delta \frac{\partial^n u(x,t)}{\partial t^n} = -D(t) \frac{\partial u(0,t)}{\partial x}
\]

Then, subtracting eq. (7) from eq. (8) and integrating the result from 0 to \(\delta\) we get [24, 25]:

\[
\int_0^\delta \left( \int_0^\delta \frac{\partial^n u(x,t)}{\partial t^n} \, dx \right) \, dx = D(t) u(0,t)
\]

Therefore, the right-side of eq. (9) depends on the boundary condition at \(x = 0\) and, in contrast to eq. (6b), is independent of the approximation of the gradient at this point.

**Assumed profile, spatial integration, and the penetration depth**

The solution in this work uses an assumed parabolic profile with unspecified exponent [23-27]:

\[
u_a(x/\delta) = (1 - x/\delta)^n, \quad 0 \leq x \leq \delta, \quad n > 0.
\]

The profile satisfies the conditions in eq. (5a,b,c) and forms two zones: \(u_a(x) > 0\) for \(x < \delta\) and \(u_a(x) = 0\) for \(x \geq \delta\). The solution based on this assumed profile requires \(n > 0\) but dependent on the fractional order \(\mu\) and the exponent \(\beta\). Now, we turn on the integration of the left-hand side of eq. (9) replacing \(u(x,t)\) by \(u_a(x,\delta)\), [24, 25]:

\[
\int_0^\delta \left( \int_0^\delta \frac{\partial^n u(x,t)}{\partial t^n} \left(1 - \frac{x}{\delta}\right)^n \right) \, dx
\]

Now, with \(D(t) = D_0 t^\beta\) and \(u(0,t) = 1\) we get:

\[
\frac{\partial^n \delta^2}{\partial t^n} = D_0 N t^\beta, \quad N = (n+1)(n+2)
\]

The Laplace transform of eq. (11a) is \(\delta^2(s) = D_0 N \Gamma(1 + \beta) s^{-(1+\beta+n)}\). Then, the inverse Laplace transforms yields:

\[
\delta^2(t) = (D_0 t^{\mu+\beta}) \int \frac{\Gamma(1+\beta)}{\Gamma[1+(\mu+\beta)]} \, \Gamma(1 + \beta)
\]

For \(\beta = 0\) the solution of eq. (12b) reduces to DIM solution [24, 25] of the time-fractional subdiffusion equation, namely: \(\delta_{\mu,\beta-0}(t) = (D_0 t^{\mu+\beta})^{1/2} [N/\Gamma(1 + \mu)]^{1/2}\). Further, for \(\mu = 1\) and \(\beta = 0\) we get \(G(1,0) = 1\) and the solution of eq. (12b) reduces to the classical DIM solution of the integer-order diffusion equation, see comments in [24, 25]. In the special case when \(\mu + \beta = 1\) and \(G(\mu, \beta)\) reduces to \(G(\mu + \beta = 1) = \Gamma(1 + \beta)\) and \(\delta_{\mu,\beta-1}(t) = (D_0 t)^{1/2} [N \Gamma(1 + \beta)]^{1/2}\) depends only on \(\beta\).

Hence, the front propagates with a speed proportional to \(t^{\mu+\beta/2}\) and the principle questions arising from this, irrespective to the initial stipulation that eq. (1a) is a subdiffusion equation are: (1) how the speed of the front depends on the mutual effects of the exponent \(\beta\) and
the fractional order $\mu'$, and (2) are there restrictions on the exponent $\beta$ taking into account that subdiffusion transport is the problem at issue?

**The speed of the penetration front and restrictions on the exponent $\beta$**

Since, the values of $\mu$ and $\beta$ define the behaviour of the diffusion processes it is important to explain clearly how they affect the solution and to delineate the two principle regimes of diffusion mentioned earlier. Hence, the front spread is proportional to $t^{\mu + \beta/2}$ with $0 < \mu < 1$ and therefore we have to define the range of variation of $\beta$.

First, since $\delta$ is a physically defined length of the diffusion process and it should be positive. Therefore, this corresponds to the condition $\mu + \beta > 0$ which can be satisfied for any $\beta > 0$. With, $d\delta/dr = [(\mu + \beta)/2] t^{\mu + \beta/2 - 1}$ the condition $d\delta/dr > 0$ requires $\mu + \beta > 2 \Rightarrow \beta > 2 - \mu$ and taking into account the range of $\mu$ the condition is satisfied for $\beta > 1$. Otherwise, the decelerating front exists for $\mu + \beta < 2$ and therefore this condition and the general one ($\mu + \beta > 0$) are satisfied for $\beta < 1$.

Further, with $0 < \mu + \beta < 1$ we have a subdiffusive regime while for $\mu + \beta > 1$ the transport is superdiffusive. In the light of the previous comments about the effect of the value of $\beta$ on the rate of the penetration front, the condition of accelerating front is $\mu + \beta > 2 \Rightarrow \beta > 2 - \mu$ which corresponds to the superdiffusion regime. In contrary, when the front is decelerating, that is $\mu + \beta < 2 \Rightarrow \beta < 2 - \mu \Rightarrow \beta < 1$ we have a transition from subdiffusion ($\mu + \beta < 1$) to superdiffusion transport ($\mu + \beta > 1$). The special case $\mu + \beta = 0 \Rightarrow \beta = -\mu$ corresponds to $\delta = \text{const.}$ and $d\delta/dr = 0$, and the stationary regime [21]. For $\mu + \beta = 1$, we obtain $\delta \equiv t^{1/2}$, that is the classic Fick’s diffusion law corresponding to the Gaussian process.

It is worthy to note that $\beta > 0$ means a diffusion coefficient increasing in time, while for $\beta < 0$ the diffusion coefficient is decreasing in time. The former case could be termed as fast diffusion, while the second one is slow diffusion. The slow diffusion with $\beta < 0$, the case solved in [21], results in $\delta \equiv t^{\mu - \beta/2}$ and $d\delta/dr = [(\mu - \beta)/2] t^{\mu - \beta/2 - 1}$. The condition $\beta < 0$ assures the front propagating in one direction that is physically motivated. In addition, we have: for $\mu - \beta - 2 > 0$ $\Rightarrow \beta < 2 - \mu$ with accelerating front, while the condition $\mu - \beta - 2 < 0$ $\Rightarrow \beta > 2 - \mu$ corresponds to the decelerating case. Both cases require $\beta > 1$ but taking into account that $0 < \mu < 1$ we have the condition $\mu - \beta - 2 < 0$ and a decelerating penetration which is a physically adequate estimation for a slow diffusion process. In addition, with respect to the mean squared displacement (see the further in this article) we have $(x^2) \equiv \delta = t^{\mu - \beta}$ and the slow subdiffusion would exist for $\mu - \beta < 1$ that is, the general condition $\mu > \beta$ previously established is satisfied. The opposite requirement $\mu - \beta > 1 \Rightarrow \mu > 1 + \beta$ does not satisfy the general condition $0 < \mu < 1$ and therefore the superdiffusion transport with $\beta < 0$ is impossible.

In this article we restrict the analysis only for the case $\beta > 0$ since the problems with $\beta < 0$ is out of the scope of the present study and draws future studies.

**Approximate profile and related issues**

**Approximate profile**

After the development of the penetration depths the approximate solution can be expressed:

$$u_a = \left(1 - \frac{x}{\sqrt{D_0 t^{\mu + \beta}}} \sqrt{NG(\mu, \beta)} \right) = \left(1 - \frac{\xi}{N_x(n, \mu, \beta)} \right)^n \quad (13)$$
\[ N_p = \sqrt{NG(\mu, \beta)} = \sqrt{(n+1)(n+2) \frac{\Gamma(1+\beta)}{\Gamma[1+(\mu+\beta)]}} \] (14)

This approximate solution defines in a natural way the similarity variable:

\[ \xi = \frac{x}{\sqrt{D_{\delta \gamma} t^{\mu+\beta}}} \]

**Mean square displacement**

Following the work of Fa a Lenzi [21] the mean square displacement (the second moment of the distribution) is eq. (15a) while with the assumed profile used in the approximate integral-balance solution (DIM solution) one obtains in eq (15b):

\[ \langle x^2 \rangle_{FL} = \frac{2D_0 \Gamma(1+\beta)}{\Gamma(1+\mu+\beta)} t^{\mu+\beta}, \quad \langle x^2 \rangle_{DIM} = \delta^2 = \frac{D_0 \Gamma(1+\beta)}{\Gamma(1+\mu+\beta)} t^{\mu+\beta} \] (15a,b)

The factor 2 in eq. (15a) appears from the integration in the interval \((-\infty, + \infty)\) [21], while the integration interval \((0, + \infty)\) is equivalent to \((0, \delta)\), in the framework of the present study. However, if the integration is performed from \((-\delta, + \delta)\) then the mathematical result is eq. (15a), but it is relevant to the case when there is a point source at \(x = 0\). Following the results of the analysis of the penetration speed and the comments in [21], the mean square displacement in eq. (15b) clearly states that subdiffusive processes arise when \(0 < \mu + \beta < 1\) and superdiffusion occurs for \(\mu + \beta > 1\).

**The first passage time: analysis from different viewpoints**

If we consider a domain with a finite length, \(L\), that is \(0 \leq x \leq L\) with boundary conditions \(u(L,t) = 0\) and \(u_t(L,t) = 0\) we may define the time required the penetration depth \(\delta(t)\) to reach the boundary \(x = L\). From eq. (12a) with \(\delta(t) = L\) we get:

\[ L^2 = \left( D_{\delta \gamma} t^{\mu+\beta} \right) NG(\mu, \beta) \Rightarrow t_L = L^{2\mu+\beta} \frac{1}{D_0 NG(\mu, \beta)} \left[ 1 - \frac{1}{(\mu+\beta)^{\mu+\beta}} \right] \] (16)

This is a practical solution coming from the integration of the distribution in the interval \((–\infty, + \infty)\) \([21]\), while the integration interval \((0, + \infty)\) is equivalent to \((0, \delta)\), in the framework of the present study. However, if the integration is performed from \((-\delta, + \delta)\) then the mathematical result is eq. (15a), but it is relevant to the case when there is a point source at \(x = 0\). Following the results of the analysis of the penetration speed and the comments in [21], the mean square displacement in eq. (15b) clearly states that subdiffusive processes arise when \(0 < \mu + \beta < 1\) and superdiffusion occurs for \(\mu + \beta > 1\).

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This is a practical solution coming from the integer-order solution developed by the integral-balance solutions corresponding to the so-called pre-heating period [22, 23].

Further we continue with the classical definition of the FPT distribution [21, 29-31] is \(F(t) = -df/dt\), where \(f = \int_0^L u(x,t)dx\). Assuming for the purpose of the following analysis that \(L = \delta(t)\) since no diffusion occurs for \(x > \delta(t)\). Hence, with the approximate solution (13) and the expression (12b) about \(\delta(t)\) we have:

\[ F_\delta (t) = -\frac{d}{dt} \left[ \frac{\delta(t)}{\delta(t)} \right]^n dx = -\frac{d}{dt} \left[ \frac{\delta(t)}{n+1} \right] = t^{\frac{\mu+\beta}{2}} \frac{\mu+\beta}{2} \sqrt{D_0} \frac{n+2}{n+1} G(\mu, \beta) \] (17a,b,c)

From the conditions about the existence of the subdiffusion regime previously commented we have the condition \(\mu + \beta > 2 > 0 \Rightarrow \beta > 2 - \mu\) corresponds to acceleration penetration front and superdiffusion. The stationary solution with a constant speed of \(\delta(t)\) for \(\mu + \beta = 0\) leads to \(F_\delta (t) = 0\). Otherwise, when \(\mu + \beta < 2 < 0 \Rightarrow \beta < 2 - \mu \Rightarrow \beta < 1\) we have subdiffusive transport. Moreover, the mean FPT (MFPT) is defined with \(\delta(t) = L\) [30, 31]:

\[ F_\delta (t) = -\frac{d}{dt} \left[ \frac{\delta(t)}{\delta(t)} \right]^n dx = -\frac{d}{dt} \left[ \frac{\delta(t)}{n+1} \right] = t^{\frac{\mu+\beta}{2}} \frac{\mu+\beta}{2} \sqrt{D_0} \frac{n+2}{n+1} G(\mu, \beta) \] (17a,b,c)
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\[ T_{\text{FPT}} = \int_0^\infty \int_0^1 u(x,t) \, dx \, dt \Rightarrow T_{\text{FPT}(\infty)} = \sqrt{D_0} \left[ \frac{1}{(n+1)N_p} \left( \frac{\mu \beta}{2} \right)^{1/2} \right] \rightarrow \infty \] (18a,b)

Therefore, the estimation of MFPT developed on the basis of the approximate integral-balance solution (DIM) is consistent with the results of Fa and Lenzi \[21, 30, 31\] and the comments of Yuste and Lindenberg \[32\], to be exact: the MFPT for the subdiffusion transport is infinite.

Refinement of the approximate solution

Residual function and approximation of the time-fractional derivative

The exponent \( n \) of the assumed profile is the only parameter which can not be defined by the boundary conditions at \( x = 0 \) and \((5a,b,c) \[22-27\] and therefore we have to impose additional constraints. The approach used here minimises the residual function of eq. (1a), precisely the mean squared error of approximation over the penetration depth \( \partial(t) \) as a procedure allowing to calculating the optimal value of the exponent \( n \). In this context, the residual function of eq. (1a) is:

\[ R[u_\times(x,t)] = \frac{\partial^n u_\times(x,t)}{\partial t^n} - D(t) \frac{\partial^2 u_\times(x,t)}{\partial x^2} \] (19)

Now, the problem is: how the time-fractional derivative \( \partial^n u_\times(x,t)/\partial t^n \) of the approximate profile can be expressed in a way easy for calculations? To resolve this problem we will apply the technique developed in \[25-27\]. Let us express the assumed profile as \( V(\eta) = (1-\eta)^n \) where \( 0 \leq \eta = x/\partial \leq 1 \). Moreover, the dimensionless profile can be approximated as a converging series \( V_\times(\eta) \approx \sum_{k=0}^N b_k \eta^k \), where \( 0 \leq \mu \leq 1 \), see \[25-27\] for details. Therefore, the residual function can be approximated:

\[ R[u_\times(x,t)] = R_{\text{RL}}(\eta, \mu, \beta, t) = \left[ \text{RL} \left( \sum_{k=0}^N b_k \eta^k \right) - D(t) \frac{\partial^2 u_\times(x,t)}{\partial x^2} \right] \] (20)

Taking into account that \( \eta = \xi/N_p \), see eq. (14), and \( \xi = [x/(D_\times)^{1/2}]t^{-\mu/2} \) all terms in \( V_\times(\eta) \approx \sum_{k=0}^N b_k \eta^k \) are power-law functions of time and can be easily differentiated:

\[ \text{RL} D_\times^\mu \left( b_k t^\lambda \right) = b_k \left[ \Gamma(1+\lambda)/\Gamma(1+\lambda-\mu) \right] t^{k-\mu}, \quad \lambda = -k(\mu + \beta)/2 \] (21a,b)

Consequently, the time-fractional derivative of order \( \mu \) from the series \( V_\times(\eta) \approx \sum_{k=0}^N b_k \eta^k \) is \[25\]:

\[ \text{RL} D_\times^\mu V(\eta) \approx \sum_{k=0}^N c_k \eta^k t^{-\mu} \approx \Phi(n, \eta, \mu) t^{-\mu}, \quad \Phi(n, \eta, \mu, \beta) = \sum_{k=0}^N c_k \eta^k \] (22a,b)

\[ c_k = b_k \left[ \Gamma(1+k)/\Gamma(1+k-\mu) \right] \] (23)

Thus, the residual function (19) can be expressed:
\[ R_{RL}(\eta, \mu, \beta, t) \approx \frac{1}{t^{\mu}} \left[ R_L(\eta) - \frac{n(n-1)(1-\eta)^{\nu-2}}{N(\eta, \mu, \beta)} \right] \quad (24) \]

With the Caputo derivative applied to \( V_a(\eta) \approx \sum_{k=0}^{\infty} b_k \eta^k \) we have:
\[ r_{RL}D^\alpha V_a(\eta) - c D^\alpha V_a(\eta) \]
\[ = R_L(\eta) - \frac{n(n-1)}{D_0^\alpha} \left( 1 - \frac{\eta}{\delta} \right)^{\nu-2} \quad (25) \]

The residual function has the same construction as the one presented by eq. (24).

**Constraints on the exponent \( n \) at the boundaries**

With the assumed profile \( u_a(x, \delta) = (1 - x/\delta)^n \) and \( u(0, t) = 1 \) the residual function with \( r_{RL}D^\alpha u_a(x, t) \) (see [25] for details and comments for the similar case with \( D(t) = D_0 \) const.) is:
\[ r_{RL}R(\xi, \mu, \beta) = \frac{1}{(1-\mu)} \left[ \frac{d}{dt} \left( \left( 1 - \frac{x}{\delta} \right)^n \right) \right] - D(t) n(n-1) \left( 1 - \frac{x}{\delta} \right)^{\nu-2} \quad (26) \]

The conditions (5b,c) i.e. \( u_a(\delta, t) = 0 \) and \( \partial u_a(\delta, t)/\partial x = 0 \) lead to, see eq. (26):
\[ r_{RL}R(\delta, t) = \lim_{x \to \delta} r_{RL}R(\delta, t) = - n(n-1) \frac{1}{\delta^\nu} \left( 1 - \frac{\xi}{\delta} \right)^{\nu-2} \quad (27) \]

Hence, the condition (27) is satisfied at \( x = \delta \) when \( n > 2 \).

Further, at \( x = 0 \) with \( u_a(0, t) = 1 \) we have:
\[ r_{RL}R(0, t) = \frac{1}{(1-\mu)} - D_0 t^\beta \frac{n(n-1)}{D_0} \frac{1}{NG(\mu, \beta)} \quad (28) \]

Setting \( r_{RL}R(0, t) = 0 \) we get the following equation relating \( n(x = 0) \) with \( \mu \) and \( \beta \):
\[ \frac{(n+1)(n+2)G(\mu, \beta)}{(1-\mu)NG(\mu, \beta)} - n(n-1)\Gamma(1-\mu) = 0 \quad (29) \]

From the denominator in eq. (29) it is hard to establish exact values of \( n \) because we have simultaneous effects of \( \mu \) and \( \beta \).

**Optimal exponents of the approximate solutions**

Applying the Langford criterion [33] for the integral-balance method we need:
\[ E_{RL}(n, \mu, \beta, t) = \int_0^\delta \left[ R_{RL}(\eta, \mu, \beta, t) \right]^2 \, dx \rightarrow \min \quad (30) \]

In terms of the dimensionless variable, \( \xi \), the squared error function (30) can be expressed as a ratio decaying in time with a rate \( t^{2\nu} \), namely \( E_{RL}(n, \alpha, \beta, t) = e_{RL}(n, \mu, \beta)/t^{2\nu} \). Here, \( e_{RL}(n, \mu, \beta) \) is time-independent function defined by:
The estimation of the optimal \( n \), through minimization of \( e_L(n, \mu, \beta) \) was carried out in accordance with the procedure explained in [26, 27] where it was established that an expansion up to nine terms of the series is enough to calculate \( \Phi(n, \eta, \mu, \beta) \) and evaluate the integral in eq. (31) by help of MAPLE, for instance. The optimal exponents determined with this technology are summarized in tab. 1. Regarding the values of the optimal exponents summarized in tab. 1 reveal some principal features:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta \rightarrow 0 )</th>
<th>0.0*</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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<tr>
<td>( E_L \cdot 10^3 )</td>
<td>3.961</td>
<td>2.840</td>
<td>1.525</td>
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<td>0.789</td>
<td>1.972</td>
<td>2.070</td>
<td>0.141</td>
<td>2.837</td>
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<tr>
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<td>3.470</td>
<td>3.736</td>
<td>3.908</td>
<td>3.969</td>
<td>4.074</td>
<td>4.312</td>
<td>3.917</td>
<td>4.894</td>
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<tr>
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<td>2.861</td>
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<td>0.653</td>
<td>8.211</td>
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</tr>
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<td>2.894</td>
<td>3.871</td>
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<td>4.390</td>
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<td>1.543</td>
<td>5.253</td>
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<td>0.324</td>
<td>1.334</td>
<td>2.264</td>
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*established in [26]; Bold (at the diagonal of the table): the case \( \mu + \beta = 1 \)

- The exponent of the approximate solution in the case of subdiffusive regime \((\mu + \beta < 1)\) decreases with increase in \( \mu \) if \( \beta \) is stipulated. Alternatively, when \( \mu \) is stipulated, then the optimal exponent increases with increase in the value of \( \beta \).
- The diagonal of the tab. 1 is formed by cells (in bold) containing the optimal exponents in the case when \( \mu + \beta = 1 \). In this special case, we have \( \tilde{\sigma}^2 \equiv t \) and the optimal exponents decrease from the top right corner \((\mu = 0.1, \beta = 0.9)\) to the left bottom corner \((\mu = 0.9, \beta = 0.1)\). The diagonal divides the data summarized in tab. 1 into two groups: subdiffusive group (LG) to the left of the diagonal and superdiffusive group (RG) to the right of the diagonal.
- For the LG, the general behaviour is that mentioned in point 1. This is well demonstrated for \( \mu = 0.1 \) and \( 0.1 \leq \beta \leq 0.9 \) with increase in \( \mu \) the right boundaries of the LG is defined by diagonal of the table (the case \( \mu + \beta = 1 \)). It is worthy to note, that close to the diagonal there is anomalous behaviour of the exponents which become close to that when \( \mu + \beta = 1 \). The transition through the diagonal \( \mu + \beta = 1 \) is demonstrated by a similar behaviour but after that the values of the optimal exponents increase with increase in \( \beta \). This transient be-
haviour of the exponents of the profile could be related to change in the speed of the front $d\delta/dt$, namely: from decelerating front (subdiffusion) to accelerating front (superdiffusion).

The behaviour of the optimal exponents previously commented can be attributed the factor $G(\mu, \beta)$ in the penetration depth, see eqs. (12c) and (14). The $G(\mu, \beta)$ varies in a narrow range $0.5 \leq G(\mu, \beta) \leq 1$, fig. 1, but its behaviour is strongly non-linear.

**Numerical experiments**

The approximate solutions are presented graphically in fig. 2. For $\mu = 0.1$, fig. 2(a), we have entirely subdiffusive profiles while for $\beta = 0.9$, fig. 2(b), the distributions correspond to the superdiffusive process. In these limiting cases as well in the intermediate situations,
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figs. 2(b), and 2(c), the arrangement of the curves from right to the left strongly follows the increase in the value of $\beta$. This arrangement could be explained by the effect of $\beta$ on the derivative $d u / d \xi$, namely:

- the derivative $du/d\xi = -(n/N_p)(1 - \xi/N_p)^{n-1}$ is a function of $\xi$ since for given $\mu$ and $\beta$, the denominator $N_p$ is a constant. However, as it was demonstrated the exponent $n$ increases with increase in $\beta$, and
- hence, if $\xi$ is fixed (a certain point of the profile) as a result of the increase in $\beta$, and the numerical value of $N_p$ and the difference $(1 - \xi/N_p)$ will increase too. Consequently, the higher value of $\beta$, the higher value of the derivative $du/d\xi$ (absolute value, since really $du/d\xi < 0$) and graphically the curves corresponding to large $\beta$ are more steep and located to the left of the profiles corresponding to small $\beta$.

The simultaneous effect of $\mu$ and $\beta$ can be visually demonstrated by the 3-D plots for fixed values of the dimensionless approximate profiles, fig. 3: close to the middle of the profile, fig. 3(a), and at for large $\xi$, figs. 3(b) and 3(c). The behaviour of the surfaces in fig. 3 resembles that of the function $G(\mu, \beta)$, fig. 1. In fact the approximate solution $u_a$ can be expressed as $u_a(\xi) = \{1 - k_o/G(\mu, \beta)\}^{1/2}$ where $k_o = \xi^2/N$. For the sake of the simplicity the plots in fig. 3 are developed for stipulated value $n = 4$.

![Figure 3. The 3-D approximate profiles at fixed values of $\xi$, as function of $\mu$ and $\beta$.](image)

The numerical results clearly demonstrate the general behaviour of the approximate solution developed by the integral-balance approach. The upper limit of $\beta$ was chosen $\beta = 0.9$ since there is no reason to use large values where in general the regime will be superdiffusive.

The integral-balance method demonstrates its feasibility in solution of a subdiffusion equation with a time-dependent of a power-law type diffusion coefficient. This study is part of a large project toward application of integral balance methods and developing approximate physically motivated solution to non-linear diffusion models as it was already done in [24-27, 34, 35].

Conclusions

The integral balance approach to the time-fractional diffusion equation with a time-dependent diffusion power-law coefficient $D(t) = D_0 t^\beta$ confirms the analytical results of Fa and Lenzi [21] that its second moment is of power-law type with exponent $\mu + \beta$. The solution can present subdiffusive ($0 < \mu + \beta < 1$), superdiffusive ($\mu + \beta > 1$), and Gaussian transport ($\mu + \beta = 1$).

The solution was performed in the infinite domain but with the integral-balance approach a front propagation of the solution is defined. This allows investigating the principle
problem of the FPT in infinite domain. The results confirm the principle statement [21, 31] that in case of subdiffusive regime the FPT is infinite.

References


