THE FOURIER-YANG INTEGRAL TRANSFORM FOR SOLVING THE 1-D HEAT DIFFUSION EQUATION

by

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A new Fourier-like integral transform (called the Fourier-Yang integral transform)

\[ S[\Lambda(t)] = \int_{-\infty}^{\infty} \Lambda(t)e^{-i\omega t}dt \]

is considered to find the fundamental solutions of the 1-D heat diffusion equation in the different initial conditions.

Key words: fundamental solution, heat equation, diffusion equation, Fourier-Yang integral transform, Fourier-like integral transform

Introduction

The PDE in the heat transfer problems are the important topics for scientists and engineers to explore the heat transport in the solid, liquid and gas [1-4]. The heat diffusion equation is one of the interesting PDE for describe the heat transfer theory [5-7] and the diffusion flow in metamorphic rocks [8, 9]. With the aid of the (non-local and local) fractional calculus, the heat diffusion equation can be generalized to fractional diffusion equations [10-12] and local fractional diffusion equations [13-15].

In order to find the solutions for the heat diffusion equations, many technologies, such as the Laplace-like integral transform [5], finite integral transform [16], homology [17], variational iteration [18], alternating-direction implicit [19], immersed interface [20], and the Laplace-like integral transform [21] methods, were developed.

A new Fourier-like integral transform (called the Fourier-Yang integral transform), proposed by Yang [22], was considered to solve the steady heat transfer problem. More integral transforms for solving the heat transfer problems were considered in [23-25]. The aim of the present manuscript is to present the properties of this integral transform and a new application to find the fundamental solution for a 1-D heat diffusion equation.

The Fourier-Yang integral transform

In this section, we introduce the concepts of Fourier and Fourier-Yang integral transforms, and properties of the Fourier-Yang integral transform.

The Fourier integral transform of the function $\Phi(t)$ is given [23]:

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\[\Phi(\theta) = \varphi [\Phi(t)] = \int_{-\infty}^{\infty} \Phi(t) e^{-j\theta t} \, dt\] (1)

where \(\varphi\) is the Fourier integral transform operator.

The inverse Fourier integral transform operator of eq. (4) is written [23]:

\[\Phi(t) = \varphi^{-1} [\Phi(\theta)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\theta) e^{j\theta t} \, d\theta\] (2)

where \(\varphi^{-1}\) is the inverse Fourier integral transform operator.

The Fourier integral formula is given [23]:

\[\Phi(t) = \varphi^{-1} [\Phi(\theta)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\theta) e^{j\theta t} \, e^{j\rho t} \, d\theta \, d\rho\] (3)

The new Fourier-Yang integral transform of the function \(\Lambda(t)\) is given [22]:

\[\Lambda(\varepsilon) = \mathbb{S} [\Lambda(t)] = \varepsilon \int_{-\infty}^{\infty} \Lambda(t) e^{-j\varepsilon t} \, dt\] (4)

where \(\mathbb{S}\) is the new Fourier-Yang integral transform operator.

The inverse Fourier-Yang integral transform operator is defined [22]:

\[\Lambda(t) = \mathbb{S}^{-1} [\Lambda(\varepsilon)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Lambda(\varepsilon) e^{j\varepsilon t} \, d\varepsilon\] (5)

where \(\mathbb{S}^{-1}\) is the inverse Fourier-Yang integral transform operator.

The Fourier-Yang integral formula is given [22]:

\[\Lambda(t) = \mathbb{S}^{-1} [\Lambda(\varepsilon)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(t) e^{j\varepsilon t} \, e^{-j\rho t} \, d\varepsilon \, d\rho\] (6)

Taking \(\sigma = j\varepsilon\), we obtain the Laplace-Carson integral transform of the function \(\Omega(t)\) [24]:

\[\Omega(\gamma) = \mathcal{R} [\Omega(t)] = \gamma \int_{0}^{\infty} \Pi(t) e^{-\gamma t} \, dt\] (7)

where \(\mathcal{R}\) is the Laplace-Carson integral transform operator.

Similarly, the inverse Laplace-Carson integral transform operator is presented [24]:

\[\Omega(t) = \mathcal{R}^{-1} [\Omega(\gamma)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{\Omega(\gamma)}{\gamma} e^{\gamma t} \, d\gamma\] (8)

The properties of the Fourier-Yang integral transform operator are as follows [22].

(T1) If \(\Lambda(\theta) = \varphi [\Lambda(t)]\) and \(\Lambda(\varepsilon) = \mathbb{S} [\Lambda(t)]\), then we have:

\[\Lambda(\theta) = \frac{1}{\varepsilon} \Lambda(\varepsilon)\quad \text{and} \quad \Lambda(\theta) = \varepsilon \Lambda(\varepsilon)\] (9)

(T2) If \(\Lambda(t) = e^{-\alpha t} \varphi(t)\), where \(\varphi(t)\) is the Heaviside unit step function, then we have:
\[ \Lambda(\varepsilon) = \frac{e}{a + je} \]  

where \( a \) is a constant.

(T3) If \( \Lambda(t) = \delta(t) \), where \( \delta(t) \) represents the Dirac function, then we have:

\[ \Lambda(\varepsilon) = \varepsilon \]  

(T4) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \), then we have:

\[ \mathbb{S}[\Lambda(t-a)] = e^{-j\varepsilon a} \Lambda(\varepsilon) \]  

(T5) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \), then we have:

\[ \mathbb{S}\left[ \frac{d\Lambda(t)}{dt} \right] = j\varepsilon \Lambda(\varepsilon) \]  

(T6) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \), then we have:

\[ \mathbb{S}\left[ \frac{d^2\Lambda(t)}{dt^2} \right] = -\varepsilon^2 \Lambda(\varepsilon) \]  

(T7) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \) and \( \Theta(\varepsilon) = \mathbb{S}[\Theta(t)] \), then we have:

\[ \mathbb{S}\left[ \Lambda(t) + \Theta(t) \right] = \Lambda(\varepsilon) + \Theta(\varepsilon) \]  

(T8) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \) and \( \Theta(\varepsilon) = \mathbb{S}[\Theta(t)] \), then we have:

\[ \mathbb{S}\left[ \int_{-\infty}^{\tau} \Lambda(t-\tau)\Theta(\tau)d\tau \right] = \frac{1}{\varepsilon} \Lambda(\varepsilon)\Theta(\varepsilon) \]  

(T9) If \( \Lambda(\varepsilon) = \mathbb{S}[\Lambda(t)] \), then we have:

\[ \mathbb{S}\left[ \int_{-\infty}^{\tau} \Lambda(t)d\tau \right] = \frac{1}{j\varepsilon} \Lambda(\varepsilon) \]  

(T10) If \( \Lambda(t) = be^{-at^2} \), where \( a > 0 \), then we have:

\[ \Lambda(\varepsilon) = \frac{be}{\sqrt{\pi a}} e^{\frac{\varepsilon^2}{4a}} \]  

Proof. We have, by the definition of the Fourier-Yang integral transform, that:

\[ \Lambda(\varepsilon) = e \int_{-\infty}^{\infty} be^{-at^2} e^{-j\varepsilon t} dt = e \int_{-\infty}^{\infty} be^{-\left(\frac{t^2}{2at} + \frac{\varepsilon^2}{4a}\right)} dt = e \int_{-\infty}^{\infty} be^{-\frac{\varepsilon^2}{4a}} dt = \frac{be}{\sqrt{\pi a}} e^{\frac{\varepsilon^2}{4a}} \]  

where

\[ \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} \]
The fundamental solution for the 1-D heat diffusion equation

In this section, we use the Fourier-Yang integral transform to solve a 1-D heat diffusion equation with the different initial conditions.

We now consider the initial value problem for a 1-D heat diffusion equation without source or sinks [23]:

$$\frac{\partial \Lambda(x,t)}{\partial t} = \psi \frac{\partial^2 \Lambda(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 < t$$  \hspace{1cm} (25)

where $\psi$ is the diffusivity constant with the initial condition:

$$\Lambda(x,0) = g(x), \quad -\infty < x < \infty$$  \hspace{1cm} (26)

We find the Fourier-Yang integral transform for this problem with respect to the space variable $x$.

Let us consider the following equations:

$$\mathcal{F} \left[ \frac{\partial \Lambda(x,t)}{\partial t} \right] = \mathcal{F} \left[ \frac{\partial \Lambda(x,t)}{\partial t} \right] e^{-j\omega t} \text{d}x = \frac{\partial \Lambda(\omega,t)}{\partial t}$$  \hspace{1cm} (27)
Substituting eqs. (27) and (28) into eq. (25), we have:
\[
\frac{d \Lambda(\varepsilon,t)}{dt} + \varepsilon^2 \Lambda(\varepsilon) = 0, \quad 0 < t
\] (29)

where
\[
\Lambda(\varepsilon,0) = \varepsilon g(\varepsilon)
\]
(30)

Finding the solution of eq. (27), we have:
\[
\Lambda(\varepsilon,t) = \varepsilon g(\varepsilon)e^{-\varepsilon^2 t}
\]
(31)

Making use of the inverse Fourier-Yang integral transform, we get:
\[
\Lambda(x,t) = \mathcal{S}^{-1}[\Lambda(\varepsilon,t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon g(\varepsilon)e^{-\varepsilon^2 t}}{\varepsilon} e^{j\varepsilon x} d\varepsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\varepsilon)e^{j\varepsilon x - \varepsilon^2 t} d\varepsilon
\]

From eq. (16), we have:
\[
\mathcal{S}\left[\int_{-\infty}^{\infty} \Lambda(x-\tau,t)\Theta(\tau,t)d\tau\right] = \frac{1}{\varepsilon} \Lambda(\varepsilon,t)\Theta(\varepsilon,t)
\]
(32)

which leads to:
\[
\mathcal{S}^{-1}\left[\frac{1}{\varepsilon} \Lambda(\varepsilon,t)\Theta(\varepsilon,t)\right] = \int_{-\infty}^{\infty} \Lambda(x-\tau,t)\Theta(\tau,t)d\tau
\]
(33)

In view of eq. (33), we have:
\[
\Lambda(x,t) = \int_{-\infty}^{\infty} g(x-\tau,t)\Theta(\tau,t)d\tau
\]
(34)

where
\[
\Theta(\tau,t) = \mathcal{S}^{-1}\left[e^{-\varepsilon^2 \tau}\right]
\]
(35)

Thus, from eq. (23), we obtain:
\[
\Lambda(x,t) = \frac{1}{\sqrt{4\pi\psi t}} \int_{-\infty}^{\infty} g(\tau)e^{-\frac{(x-\tau)^2}{4\psi t}} d\tau
\]
(36)

This result is with agreement with the solution of the 1-D heat diffusion equation by using Fourier transform [23].

Let \(\Lambda(x,0) = g(x) = \delta(x)\) in eq. (26). Then, from eq. (36) we have:
\[
\Lambda(\varepsilon,t) = \varepsilon^2 e^{-\varepsilon^2 t}
\]
(37)

With the use of the inverse Fourier-Yang integral transform, we have:
Thus, we obtain the solution for the 1-D heat diffusion equation:

\[ \Lambda(x,t) = \frac{1}{\sqrt{4\pi \psi t}} \int_{-\infty}^{\infty} \delta(\tau)e^{-\frac{(x-\tau)^2}{4\psi \tau}} d\tau \]  

which results in:

\[ \Lambda(x,t) = \frac{1}{\sqrt{4\pi \psi t}} e^{-\frac{x^2}{4\psi t}} \]  

This result is in accordance with the solution of the 1-D heat diffusion equation by using Fourier-like transform [5].

Let \( \Lambda(x,0) = g(x) = e^{-x^2} \) in eq. (26). Then, from eq. (36) we have the solution in the Fourier-Yang integral transform:

\[ \Lambda(\epsilon,t) = e^{\frac{x^2}{\psi}} \]  

By the inverse Fourier-Yang integral transform, we have:

\[ \Lambda(x,t) = \mathcal{S}^{-1}[\Lambda(\epsilon,t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{x^2}{\psi}} e^{i\epsilon x} d\epsilon = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{x\sqrt{\pi}} e^{-\frac{x^2}{4\psi \tau}} d\epsilon \]  

which leads to:

\[ \Lambda(x,t) = \frac{1}{\sqrt{4\pi \psi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\psi \tau}} d\tau \]  

**Conclusion**

We present the new application of the Fourier-Yang integral transform to solve the initial value problem for the 1-D heat diffusion equation in this work. The fundamental solutions of this problem with the initial conditions were obtained with the use of the Fourier-Yang integral transform. The approach for solving this problem is efficient and accurate.

**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>( t )</td>
<td>time, [s]</td>
</tr>
<tr>
<td>( x )</td>
<td>space co-ordinate, [m]</td>
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</tbody>
</table>

**Greek symbols**

- \( \Lambda(x,t) \) – temperature, [K]
- \( \psi \) – diffusivity constant, [Wm\(^{-1}\)K\(^{-1}\)]

**References**


