A NEW APPROACH FOR FINDING STANDARD HEAT EQUATION AND A SPECIAL NEWELL-WHITEHEAD EQUATION

by

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Under a frame of $2 \times 2$ matrix Lie algebras, Tu and Meng [9] once established a united integrable model of the Ablowitz-Kaup-Newel-Segur (AKNS) hierarchy, the D-AKNS hierarchy, the Levi hierarchy and the TD hierarchy. Based on this idea, we introduce two block-matrix Lie algebras to present an isospectral problem, whose compatibility condition gives rise to a type of integrable hierarchy which can be reduced to the Levi hierarchy and the AKNS hierarchy, and so on. A united integrable model obtained by us in the paper is different from that given by Tu and Meng. Specially, the main result in the paper can be reduced to two new various integrable couplings of the Levi hierarchy, from which we again obtain the standard heat equation and a special Newell-Whitehead equation.

Key words: Lie algebra, TAH scheme, DS hierarchy, heat equation, Newell-Whitehead equation

Introduction

Tu [1] proposed by using $2 \times 2$ Lie algebras a scheme for generating integrable Hamiltonian hierarchies of evolution equations which was called the Tu scheme [2]. Under the frame of the Tu scheme, many interesting integrable Hamiltonian hierarchies and some corresponding properties were obtained, such as the consequences in [2-8]. Tu and Meng [9] employed the $2 \times 2$ Lie algebra:

\begin{center}
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\end{pmatrix}
\end{center}

with the commutative relations

\begin{center}
\[
[l_1,l_2] = 0, \quad [l_1,l_3] = l_3, \quad [l_2,l_3] = -l_3, \quad [l_1,l_4] = -l_4, \quad [l_2,l_4] = l_4, \quad [l_3,l_4] = l_1 - l_2
\end{center}

\begin{center}
to construct an integrable model which could be reduced to the Levi hierarchy, D-AKNS hierarchy and TD hierarchy.
\end{center}

In the paper, we want to introduce two types of block-matrix Lie algebras for which a united integrable model of the Levi hierarchy and the AKNS hierarchy is obtained. Again, the integrable model is further reduced to two different integrable couplings of the Levi hierarchy, one of them is reduced to the standard heat equation, another one can give a special

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Newell-Whitehead equation, however, the Newell-Whitehead equation is not integrable, which is an interesting fact.

Two Lie algebras

Tu [1] detailed the Lie algebras for generating the integrable Hamiltonian hierarchies. Based on this, we first present the known simple algebra which consists of the following $2 \times 2$ matrices:

\[
h_1 = l_1, \quad h_2 = l_3, \quad h_3 = l_4, \quad h_4 = l_2
\]

along with the commutative relations:

\[
[h_1, h_2] = h_1 h_2 - h_2 h_1 = h_2, \quad [h_1, h_3] = -h_3, \quad [h_1, h_4] = 0, \quad [h_2, h_3] = h_1 - h_4
\]

A loop algebra of the Lie algebra is defined:

\[
\mathcal{H} = \{ h(n), \; i = 1, 2, 3, 4; \; n \in \mathbb{Z} \}
\]

where $[h(m), h(n)] = [h_i, h_j] \lambda^{m+n}, \; m, n \in \mathbb{Z}$.

Denote:

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

From this $2 \times 2$ matrices, we introduce the following Lie algebra:

\[
G_1 = \{ f_1, \ldots, f_8 \}
\]

where

\[
f_1 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad f_2 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad f_3 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad f_4 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix}, \quad f_7 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}
\]

along with the commutative relations:

\[
[f_1, f_2] = 0, \quad [f_1, f_3] = f_3, \quad [f_1, f_4] = -f_4, \quad [f_2, f_3] = -f_3, \quad [f_2, f_4] = f_4, \quad [f_3, f_4] = f_1 - f_2
\]

\[
[i, f_5] = 0, \quad i = 1, 2, 3, 4, 6, 7, 8; \quad [f_1, f_6] = 0, \quad [f_1, f_7] = f_8, \quad [f_1, f_8] = f_7, \quad [f_2, f_6] = 0
\]

\[
[f_2, f_3] = -f_8, \quad [f_2, f_5] = -f_7, \quad [f_3, f_5] = -f_7 - f_8, \quad [f_3, f_7] = f_9, \quad [f_3, f_8] = -f_9
\]

\[
[f_4, f_6] = f_7 - f_8, \quad [f_4, f_7] = -f_6, \quad [f_4, f_8] = -f_6, \quad [f_6, f_7] = -2f_8, \quad [f_6, f_8] = 2f_7, \quad [f_7, f_8] = -2f_6
\]

Define:

\[
\tilde{G}_1 = \{ f_1(n), \ldots, f_8(n) \}
\]

where

\[
f_i(n) = f_i \lambda^n, \quad [f_i(m), f_j(n)] = [f_i, f_j] \lambda^{m+n}, \quad 1 \leq i, j \leq 8; \quad m, n \in \mathbb{Z}
\]

It is easy to see that $\tilde{G}_1$ is a loop algebra, where $f_5(n) = \tilde{G}_1$ – a pseudo-regular, and satisfies the following properties:

- $\tilde{G}_1 = \ker \text{ad} f_5(n) \oplus \text{im} \text{ad} f_5(n)$
- $\ker \text{ad} f_5(n)$ is commutative.

Generally, we have the following proposition [1]:

\[
\text{ker} \text{ad} f_5(n) = \text{im} \text{ad} f_5(n) \]

\[
\tilde{G}_1 = \text{ad} f_5(n) \]
If \( X \) is a regular element of a semisimple Lie algebra \( G \), then \( R = X \otimes \lambda \) is pseudo-regular in \( G \). In what follows, we establish another Lie algebra with the help of the above \( 2 \times 2 \) matrices:

\[ G_1 = \{g_1, \ldots, g_7\} \]

where

\[
g_i = f_i, \quad i = 1, 2, 3, 4; \quad g_5 = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix}, \quad g_7 = \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix}
\]

along with the commutative relations:

\[
[g_1, g_2] = 0, \quad [g_1, g_3] = g_3, \quad [g_1, g_4] = -g_4, \quad [g_2, g_3] = -g_3, \quad [g_2, g_4] = g_4, \quad [g_3, g_4] = g_1 - g_2
\]

\[
[g_1, g_5] = 0, \quad [g_1, g_6] = g_1, \quad [g_1, g_7] = g_6, \quad [g_2, g_5] = 0, \quad [g_2, g_6] = -g_1, \quad [g_2, g_7] = -g_6
\]

\[
[g_3, g_5] = -g_5 - g_7, \quad [g_3, g_6] = g_5, \quad [g_3, g_7] = -g_5, \quad [g_4, g_5] = g_6, \quad [g_4, g_6] = -g_7, \quad [g_4, g_7] = -g_5
\]

\[
[g_4, g_5] = -g_5, \quad [g_5, g_6] = 2g_7, \quad [g_5, g_7] = 2g_6, \quad [g_6, g_7] = -2g_5
\]

If set \( \Delta_1 = \{g_1, g_2, g_3, g_4\}, \Delta_2 = \{g_5, g_6, g_7\} \), we find that:

\[ G_2 = \Delta_1 \oplus \Delta_2 \subset \Delta_1, i = 1, 2; \quad [\Delta_1, \Delta_2] \subset \Delta_2 \quad \text{(2)} \]

A corresponding loop algebra is defined:

\[ \tilde{G}_2 = \{g_i(n), g_j(n), \ldots, g_7(n)\} \quad \text{(3)} \]

where

\[ g_i(n) = g_i \lambda^n, \quad [g_i(n), g_j(n)] = [g_i, g_j] \lambda^{m+n}, \quad 1 \leq i, j \leq 7; \quad m, n \in \mathbb{Z} \]

The subalgebras \( \Delta_1 \) and \( \Delta_2 \) are all semisimple.

**An expanding integrable model of the Levi hierarchy and some reductions**

Consider an isospectral Lax pair by the loop algebra \( \tilde{G}_1/\phi, -U \phi, \phi t = V \phi \), where:

\[
U = f_1(1) + (r - q) f_2(0) + q f_3(0) + r f_4(0) + u_1 f_5(0) + u_2 f_6(0) + u_3 f_7(0) \quad \text{(4)}
\]

\[
V = V_1 f_1(0) + V_2 f_3(0) + V_3 f_4(0) + V_4 f_2(0) + \sum_{i=6}^{8} V_i f_i(0) \quad \text{(5)}
\]

where

\[ V_i = \sum_{m=0}^{\infty} V_{i,m} \lambda^{-m} \]

Then the stationary zero curvature equation admits that:

\[
\begin{align*}
(V_{1,m})_x &= q V_{3,m} - r V_{2,m}, \quad V_{1,m+1} = -(V_{2,m})_x - (r-q) V_{3,m} - 2q V_{1,m} \\
V_{3,m} &= (V_{3,m})_x - (r-q) V_{3,m} - 2r V_{1,m}, \quad (V_{3,m})_y = -(V_{1,m})_x, \\
V_{4,m} &= -(V_{1,m})_x + (r-q+2u_1) V_{2,m} - (q+r+2u_1) V_{3,m} - u_1 V_{4,m} + u_1 V_{2,m} - 2u_1 V_{1,m} \quad \text{(6)}
\end{align*}
\]

\[
\begin{align*}
V_{7,m+1} &= -(V_{7,m})_x + (q-r+2u_1) V_{6,m} - (q-r+2u_1) V_{7,m} + u_1 V_{7,m} + u_1 V_{7,m-2} - 2u_1 V_{1,m} \\
V_{8,m+1} &= -(V_{7,m})_x + (q-r+2u_1) V_{6,m} - (q-r+2u_1) V_{7,m} + u_1 V_{7,m} + u_1 V_{7,m-2} - 2u_1 V_{1,m}
\end{align*}
\]

The eq. (6) are local solvable. If set:

\[ V_{1,0} = \alpha = \text{constant}, \quad V_{2,0} = \ldots = V_{8,0} \]
we can get from eq. (6) that:
\[ V_{1,1} = 0, \quad V_{2,1} = -2aq, \quad V_{3,1} = -2ar, \quad V_{2,2} = 2a[q_x - q(q - r)], \quad V_{3,2} = -2a[r_x + r(q - r)] \]
\[ V_{1,2} = -2aq, \quad V_{2,3} = 2a[q_x + (q - r), q + 2(q - r)q_x - (q - r)^2 q + 2q^2 r], \quad V_{7,1} = -2aq_u \]
\[ V_{8,1} = -2au, \quad V_{6,3} = 0, \quad V_{7,2} = 2a[u_{3,x} - u_2(r - q + 2u) + ru_1 - qu_1] \]

Set:
\[
V_{n}^{(a)} = \sum_{n=0}^{\infty} \left[ V_{in} f_i(-m) + V_{2n} f_1(-m) + V_{3n} f_3(-m) + V_{4n} f_2(-m) \right] + \sum_{i=0}^{\infty} V_{in} f_i(-m)
\]

then eq. (5) can be decomposed into an equivalent equation:
\[
-V_{n,x}[U, V^{(a)}] = -[U, V^{(a)}] \quad (7)
\]

By following the approach presented in [1], we can find that:
\[
-V_{n,x}[U, V^{(a)}] = V_{2n+1,1} f_1(0) - V_{3n+1,1} f_3(0) + V_{8,1} f_8(0)
\]

Take \( V^{(a)} = V^{(a)} + k f_2(0) + k f_6(0) \) a direct calculation gives:
\[
V^{(a)} - [U, V^{(a)}] = k, \quad f_1(0) - (V_{3n+1,1} + qk_1) f_3(0) + (V_{2n+1,1} + rk_k) f_6(0) -
\]

Thus, the compatibility condition of the Lax pair:
\[
\psi_x = U\psi, \quad \psi_t = V^{(a)}\psi
\]
gives rise to the following integrable hierarchy:
\[
\begin{cases}
(r - q) t = k_1, x = -V_{8,1} - u_2 + (q - r + 2u_1) u_2 \\
u_{2,1} = V_{7,1} - u_2 k_1 + (2u_2 + q + r) k_2 \\
u_{3,1} = -V_{3,1} - qu_1 + r u_1 = k_2
\end{cases} \quad (8)
\]

If set \( k_1 = V_{3n} - V_{2n} + 2V_{1n}, u_2 = u_3 = 0 \), eq. (8) reduces to the well-known Levi hierarchy:
\[
\begin{cases}
q_t = V_{2n,1} + rV_{2n} - qV_{3n} \\
r_t = V_{3n,1} + qV_{2n} - rV_{2n}
\end{cases} \quad (9)
\]

If set \( k_1 = k_2 = u_2 = u_3 = 0 \), then eq. (8) reduces to the AKNS hierarchy:
\[
\begin{cases}
q_t = -V_{2,1} \\
r_t = V_{3,1}
\end{cases} \quad (10)
\]

Therefore, eq. (8) is a united integrable model of the Levi hierarchy and the AKNS hierarchy, and it is different from the united integrable model given by Tu and Meng [9].

If take \( k_1 = V_{3n} - V_{2n} + 2V_{1n}, k_2 = -V_{6n} \), eq. (8) becomes:
\[
\begin{cases}
q_t = -V_{2,1} - q(V_{3n} - V_{2n} + 2V_{1n}), r_t = V_{3,1} + r(V_{3n} - V_{2n} + 2V_{1n}) \\
u_{2,1} = -V_{8,1} - u_2 (V_{3n} - V_{2n} + 2V_{1n}) + (r - q - 2u_1) V_{6n} \\
u_{3,1} = -V_{7,1} - u_2 (V_{3n} - V_{2n} + 2V_{1n}) - (q + r + 2u_2) V_{6n} \\
u_{4,1} = -V_{6,1}
\end{cases} \quad (11)
\]
Cases 1: \( u_1 = V_{an} = 0 \). Equation (11) presents that:
\[
\begin{align*}
q_1 &= V_{2n,x} + rV_{2n} - qV_{3n} = V_{2n,x} - V_{1n,x}, r_1 = V_{3n,x} + qV_{3n} - rV_{2n} = V_{3n,x} + V_{1n,x} \\
u_{2,j} &= V_{7n,x} + (r - q - 2u_i)V_{8n} - (u_j + u_3)V_{3n} + (u_3 - u_j)V_{2n} \\
u_{3,j} &= V_{8n,x} + (q - r - 2u_i)V_{7n} + (u_3 - u_2)V_{3n} + (u_2 - u_3)V_{2n}
\end{align*}
\]

According to the theory on integrable couplings [7, 8], eq. (12) is an integrable coupling of the Levi hierarchy (9). When \( n = 2 \), we can get the coupled part of the Levi equation:
\[
\begin{align*}
2u_{2,j} &= 2au_{3,xx} - 2c(u_2r - qu_2)_x + 2c(r - q)(qu_3 - ru_2 - ru_3 - qu_2 - u_2^2) + 2au_3(r_2 + rq - r^2)_x + 2au_3(q_x - q^2 + qr)_x \\
u_{3,j} &= 2au_{2,xx} - 2c(u_3q - u_2r)_x + 2c(q - r)(u_{1,xx} - u_2q + ru_1 - qu_1) + 2au_2(r_1 + qr - r^2) + 2au_2(q_x - q^2 + qr)
\end{align*}
\]

Specially, if set \( q = r = 0 \), eq. (13) reduces to:
\[
\begin{align*}
u_{2,j} &= 2au_{3,xx} \\
u_{3,j} &= 2au_{2,xx}
\end{align*}
\]

Take \( u_2 = u_3 = v \), eq. (14) is just right the well-known linear heat equation:
\[
\begin{align*}
u_r &= 2av_x
\end{align*}
\]

Case 2: \( u_1 \neq 0 \). Equation (11) becomes:
\[
\begin{align*}
q_1 &= V_{2n,x} - V_{1n,x}, r_1 = V_{3n,x} + V_{1n,x}, u_{1,j} = -V_{6n,x} \\
u_{2,j} &= V_{7n,x} - (q - r + 2u_i)V_{8n} - (u_j + u_3)V_{3n} + (u_3 - u_j)V_{2n} \\
u_{3,j} &= V_{8n,x} + (q - r - 2u_i)V_{7n} + (u_3 - u_2)V_{3n} + (u_2 - u_3)V_{2n}
\end{align*}
\]

When \( n = 2 \), eq. (16) reduces to:
\[
\begin{align*}
q_1 &= 2au_{3,xx} - 2c(q_x^2 - 2qr)_x, r_1 = -2aru_3 - 2c(2qr - r^2)_x \\
u_{1,j} &= -2c(qu_3 - ru_3 - ru_3 - qu_2 - u_2^2)_x \\
u_{2,j} &= 2au_{3,xx} - 2c(ru_2 - qu_2 + 2u_2u_3 - ru_3 + qu_3)_x - 2c(q - r + 2u_i)(u_{2,xx} - qu_2 + ru_3 - 2u_2u_3 - ru_3 - qu_3) + 2au_3(r_2 + q - r^2) + 2au_3(q_x - q^2 + qr)_x \\
u_{3,j} &= 2au_{2,xx} - 2c(u_3q - u_2r + 2u_2u_3 - ru_3 - qu_3)_x + 2c(q - r - 2u_i)(u_{3,xx} - u_3q + u_3q + 2u_2u_3 + ru_3 - qu_3) - 2au_2(r_1 + q - r^2) + 2au_2(q_x - q^2 + qr)
\end{align*}
\]

If set \( q = r = 0 \), eq. (17) gives:
\[
\begin{align*}
u_{1,j} &= 4au_{2,xx}u_{2,xx}, u_{2,j} = 2au_{3,xx} - 4c(u_1u_2)_x + 8au_{3,xx}^2u_3 \\
u_{3,j} &= 2au_{2,xx} - 4c(u_1u_3)_x - 8au_{2,xx}^2u_2
\end{align*}
\]

The first equation in eq. (18) is a conserved form, and later two are linear with respect to the variables \( u_3 \) and \( u_1 \), respectively:
If set \( u_2 + u_3 = u, u_2 - u_3 = v, \alpha = 1/2, \)
eq (18) becomes:
\[
\begin{align*}
  u_t &= -\frac{1}{2}(u_1 u) + 4u_1^2 v \\
  v_t &= -\frac{1}{2}(u_1 v) + 4u_1^2 u
\end{align*}
\]
where the variable \( u_1 \) satisfies that
\[
u_1,t = \frac{1}{2}(u + v)(u_1 + v).
\]

**Case 3:** \( k_1 = k_2 = 0 \). Equation (8) just reduces to an integrable coupling of the AKNS hierarchy:
\[
\begin{align*}
  q_t &= -V_{2,n+1}, q_r = V_{3,n+1} \\
  u_{2,t} &= -V_{8,n+1}, u_{3,t} = -V_{7,n+1}
\end{align*}
\]
Equation (19) can be written:
\[
\begin{align*}
  u_t &= \begin{pmatrix} q \\ r \\ u_1 \\ -V_{8,n+1} \end{pmatrix} \\
  u_{2,t} &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
  u_{3,t} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]
where
\[
J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},
L = \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \\ -2u_2\partial^{-1}q & 2u_2\partial^{-1}r \\ 2u_1\partial^{-1}q & -2u_1\partial^{-1}r \end{pmatrix}
\]
when \( u_2 = u_3 = 0 \), the above \( J \) and \( L \) reduce to the case of the AKNS hierarchy.

**Another expanding integrable model of the Levi hierarchy and some reductions**

Consider an isospectral problem by the loop algebra \( \hat{G}_2 \):
\[
\psi_x = U\psi, \psi_t = V\psi
\]
where
\[
U = g_2(1) + (r - q)g_2(0) + qg_3(0) + rg_4(0) + s_1g_5(0) + s_2g_6(0)
\]
\[
V = V_{1,1}g_1(0) + V_{2,1}g_1(0) + V_{3,1}g_1(0) + V_{4,2}g_2(0) + V_{5,2}g_2(0) + V_{6,2}g_2(0) + V_{7,2}g_2(0)
\]
where
\[
V_i = \sum_{m=0}^{\infty} V_{m,i} z^{-m}, i = 1, 2, \ldots, 7
\]
Similar to the previous discussion, we can get a recursion relation among \( V_{m,i} \):
\[
\begin{align*}
  V_{1,m,i} &= qV_{1,m-1} - rV_{2,m} = -V_{4,m,i}, V_{2,m,i} = -V_{3,m,i} - (r - q)V_{2,m} - 2qV_{1,m} \\
  V_{3,m+1,i} &= V_{3,m+1,i} - (r - q)V_{3,m} - 2rV_{1,m} \\
  V_{4,m,i} &= -(q + r + 2s_1)V_{4,m} + (q - r + 2s_2)V_{6,m} + (s_1 + s_2)V_{5,m} + (s_2 - s_1)V_{2,m} \\
  V_{5,m,i} &= -(q + r)V_{7,m} + (q - r)V_{7,m} - 2s_2V_{3,m} - 2s_1V_{1,m} \\
  V_{6,m,i} &= -(q + r)V_{6,m} - (q + r)V_{6,m} - 2s_1V_{4,m} - 2s_2V_{2,m}
\end{align*}
\]
If denote:

\[ V^{(s)} = \sum_{m=1}^{n} \left[ V_{m} g_{1}(n-m) + V_{3m} g_{3}(n-m) + V_{4m} g_{2}(n-m) + \sum_{i=3}^{m} V_{m} g_{1}(n-m) \right] + (V_{3a} - V_{3a} + 2V_{1e})g_{2}(0) \]

we can obtain:

\[ V^{(s)} - [U, V^{(s)}] = [V_{2,a+1} + q(V_{3a} - V_{2a} + 2V_{1e})]g_{1}(0) + [V_{3,a+1} + r(V_{3a} - V_{2a} + 2V_{1e})]g_{4}(0) + (V_{2a} - V_{2a} + 2V_{1e})g_{2}(0) - (V_{3a} - 2V_{2a} + 2V_{1e})g_{2}(0) - (V_{3a} - 2V_{2a} + 2V_{1e})g_{2}(0) \]

Therefore, the zero curvature equation \( U_{t} - V^{(s)} + [U, V^{(s)}] = 0 \) admits:

\[
\begin{align*}
q_{1} &= -V_{2,a+1} - q(V_{3a} - V_{2a} + 2V_{1e}), \quad r_{1} = V_{3,a+1} + r(V_{3a} - V_{2a} + 2V_{1e}) \\
q_{2} &= -V_{6,a+1} - 2s_{1}(V_{3a} - V_{2a} + 2V_{1e}) - V_{7,a+1} + (r - q)V_{6a} + (q + r + 2s_{1} + 2s_{2})V_{5a} + s_{1}V_{2a} + s_{2}V_{3a} \tag{25}
\end{align*}
\]

When \( s_{1} = s_{2} = 0 \), eq. (25) reduces to the Levi hierarchy. According to the theory on integrable couplings, eq. (25) is one integrable coupling of the Levi hierarchy, which is different from the first integrable coupling of the Levi hierarchy eq. (11). We can see this point from their reductions.

Set \( V_{i,0} = a, V_{0} = ... V_{7,0} = 0 \), we have from eq. (24):

\[
\begin{align*}
V_{1} &= -2a s_{1}, \quad V_{6} = -2a s_{1}, \quad V_{5} = 0, \quad V_{5} = 2a s_{1} - 2a(q - r)s_{1} \\
V_{7} &= 2a s_{1} - 2a(q - r)s_{2}, \quad V_{5} = 2a(-qs_{2} - rs_{1} - rs_{2} + qs_{2} - s_{1}^{2} + s_{2}^{2})...
\end{align*}
\]

When \( n = 2 \), eq. (25) reduces to:

\[
\begin{align*}
q_{1} &= 2a r_{1} - 2a(q^{2} - 2qr), \quad r_{1} = -2a r_{1} - 2a(2qr - r^{2}), \\
s_{1} &= 2a s_{1} - 2a(qs_{1} - qr)s_{1} + 2a r_{1} + 2a s_{1} - 2a(q - r^{2})s_{1} + 2a r_{1} + 2a s_{1} - 2a(q - r^{2})s_{1} + 2a r_{1} + 2a s_{1} - 2a(q - r^{2})s_{1} + 2a r_{1} + 2a s_{1} - 2a(q - r^{2})s_{1} \tag{26}
\end{align*}
\]

Equation (26) is different from eq. (17). Specially, when set \( q = r = 0 \), eq. (26) gives:

\[
\begin{align*}
q_{1} &= 2a s_{1} - 4a s_{1}(s_{1}^{2} - s_{1}^{2}) \\
s_{1} &= 2a s_{2} - 4a s_{2}(s_{2}^{2} - s_{2}^{2}) \tag{27}
\end{align*}
\]

which is various from eq. (16) which was reduced from the integrable coupling eq. (11). Hence, the integrable coupling of the Levi hierarchy eq. (25) is really different from the integrable coupling eq. (19).

We see that when \( s_{1} = s_{2} \), eq. (27) reduces to eq. (16). When set \( s_{1} = is_{2}, s_{2} = v \), eq. (27) gives:

\[ v = v_{a} + 4v^{3} \tag{28} \]
which is a special Newell-Whitehead equation.

This is an integrable equation, but the Newell-Whitehead equation:

\[ u_t = u_{xx} + u - u^3 \]  \hspace{1cm} (29)

It is not integrable. Therefore, eq. (28) could possess the similar travelling wave solutions.

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