A NEW GENERAL FRACTIONAL-ORDER DERIVATIVE WITH RABOTNOV FRACTIONAL-EXPONENTIAL KERNEL APPLIED TO MODEL THE ANOMALOUS HEAT TRANSFER

by

Xiao-Jun YANG\textsuperscript{a}, Mahmoud ABDEL-ATY\textsuperscript{b,c,d}, and Carlo CATTANI\textsuperscript{e,f}

\textsuperscript{a} State Key Laboratory for GeoMechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
\textsuperscript{b} Center for Photonics and Smart Materials (CPSM), Zewail City of Science and Technology, Zewail, Egypt
\textsuperscript{c} Mathematics Department, Faculty of Sciences, Sohag University, Sohag, Egypt
\textsuperscript{d} Applied Science University, Sitra, Kingdom of Bahrain
\textsuperscript{e} Engineering School, DEIM, University of Tuscia, Viterbo, Italy
\textsuperscript{f} Ton Duc Thang University, HCMC, Ho Chi Minh City, Vietnam

Original scientific paper
https://doi.org/10.2298/TSCI180320239Y

In this paper, we consider a general fractional-order derivataive of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function for the first time. A new general fractional-order derivataive heat transfer model is discussed in detail. The general fractional-order derivataive formula is a new mathematical tool proposed to model the anomalous behaviors in complex and power-law phenomena.

Key words: power law Rabotnov fractional-exponential function, general fractional-order derivataive, heat transfer, non-singular kernel

Introduction

The general fractional-order derivatives, where the non-singular kernels are the special functions, for more details see [1-3], such as exponential, Mittag-Leffler-Gauss, Kohlrausch-Williams-Watts, Miller-Ross, Lorenzo-Hartley, Gorenflo-Mainardi, Bessel, Mittag-Leffler, Wiman, and Prabhakar, have been applied to investigate the mathematical models in mathematical physics. The general fractional-order diffusion was reported [4]. The general-order chemical kinetics via Mittag-Leffler kernel was proposed [5]. The general fractional-order relaxation via exponential kernel was discussed [6]. The general fractional-order rheological model via Prabhakar kernel was considered [7]. The general fractional-order Burgers via Mittag-Leffler was investigated [8]. For more models via the special functions, we refer to the results for the relaxation and rheological arising in complex and power-law phenomena [1].

The Rabotnov fractional-exponential function, proposed in 1954 by Rabotnov [9], was used to describe the viscoelasticity [10, 11]. However, up to now, the general fractional-order derivative with the non-singular kernel of the Rabotnov fractional-exponential function [11] has not been developed. Motivated by the new idea, the main target of the paper is to propose the general fractional-order derivative with the non-singular kernel of the Rabotnov fraction-
al-exponential function in the sense of Liouville-Caputo type and to investigate the general fractional-order derivative heat transfer model.

**A new general fractional-order derivative with Rabotnov fractional-exponential function**

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+_0$, $\mathbb{N}$, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ be the sets of complex numbers, real numbers, non-negative real numbers, positive integers, and non-negative integers, respectively.

The Rabotnov fractional-exponential function

Let $\tau \in \mathbb{R}$, $\alpha \in \mathbb{R}^+_0$, $\lambda \in \mathbb{R}^+_0$, and $\kappa \in \mathbb{N}_0$. The Rabotnov fractional-exponential function is defined as [1, 9]:

$$\Phi_\alpha (\lambda \tau^\alpha) = \sum_{\varepsilon=0}^{\kappa - 1} \frac{\lambda^{\varepsilon} \tau^{\varepsilon (\alpha + 1)}}{\Gamma((\kappa + 1 \alpha + 1))}$$  (1)

and its Laplace transform is [1]:

$$L\{\Phi_\alpha (\lambda \tau^\alpha)\} = \frac{1}{s^{\alpha + 1}} \frac{1}{1 - \lambda s^{-(\alpha + 1)}} \left(|\lambda s^{-(\alpha + 1)}| < 1\right)$$  (2)

where the Laplace transform of the function $\phi(\tau)$ is given as [1-3]:

$$L[\phi(\tau)] := \phi(s) = \int_0^\infty e^{-s \tau} \phi(\tau) d\tau$$  (3)

with $s \in \mathbb{C}$.

**A new general fractional-order derivative with Rabotnov fractional-exponential kernel**

Let $L(a, b)$ be the set of those Lebesgue measurable functions on a finite interval $(a, b)(-\infty \leq a \leq b \leq +\infty)$, for more details, see [1].

Let $AC(a, b)$ be the space of the functions which are absolutely continuous on a finite interval $(a, b)(-\infty \leq a \leq b \leq +\infty)$, for more details, see [1].

Let $AC^0(a, b)$ be the Kolmogorov-Fomin condition, for more details, see [1].

Let $\lambda \in \mathbb{R}^+_0$. The general fractional-order integral operator via Rabotnov fractional-exponential kernel is defined:

$$\left(a \int_{\tau}^{(\alpha)} \Theta\right)(\tau) = \int_a^\tau \Phi_\alpha \left[-\lambda (\tau - t)^\alpha\right] \Theta(t) dt$$  (4)

which leads

$$\left(a \int_{0}^{(\alpha)} \Theta\right)(\tau) = \int_0^\tau \Phi_\alpha \left[-\lambda (\tau - t)^\alpha\right] \Theta(t) dt$$  (5)

where $a = 0$ and $\Theta \in L(a, b)$

$$\left(\int_{-\infty}^{(\alpha)} \Theta\right)(\tau) = \int_{-\infty}^\tau \Phi_\alpha \left[-\lambda (\tau - t)^\alpha\right] \Theta(t) dt$$  (6)

where $\Theta \in L(-\infty, b)$

$$\left(\int_{0}^{(\alpha)} \Theta\right)(\tau) = \int_0^\tau \Phi_\alpha \left[-\lambda (\tau - t)^\alpha\right] \Theta(t) dt$$  (7)

where $\Theta \in L(-\infty, b)$. 
The left-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

\[
\left( {}^{c}D_{a}^{\alpha}(\Theta) \right) = {}^{c}D_{a}^{\alpha}(\Theta) = \int_{a}^{t} \Phi_{\alpha} \left[ -\lambda (t-\tau)^{\alpha} \right] \Theta^{(\alpha)}(\tau) d\tau
\]

which can be written

\[
\left( {}^{c}D_{a}^{\alpha}(\Theta) \right) = {}^{c}D_{a}^{\alpha}(\Theta) = \int_{a}^{t} \Phi_{\alpha} \left[ -\lambda (t-\tau)^{\alpha} \right] \Theta^{(\alpha)}(\tau) d\tau
\]

where \( \Theta \in AC^{1}(a, b) \).

The right-sided general fractional-order derivative of the Liouville-Caputo type with the non-singular kernel of the Rabotnov fractional-exponential function is defined:

\[
\left( {}^{c}D_{b}^{\alpha}(\Theta) \right) = {}^{c}D_{b}^{\alpha}(\Theta) = \int_{t}^{b} \Phi_{\alpha} \left[ -\lambda (t-\tau)^{\alpha} \right] \Theta^{(\alpha)}(\tau) d\tau
\]

which can be written:

\[
\left( {}^{c}D_{b}^{\alpha}(\Theta) \right) = {}^{c}D_{b}^{\alpha}(\Theta) = \int_{t}^{b} \Phi_{\alpha} \left[ -\lambda (t-\tau)^{\alpha} \right] \Theta^{(\alpha)}(\tau) d\tau
\]

where \( \Theta \in AC^{1}(a, b) \).
General fractional-order integrals via special function

The left-sided general fractional-order integral of $\Omega(\tau)$ is defined:

$$
\left( \frac{a^{(\alpha)}}{a} \right)(\tau) = \int_{a}^{\tau} \Lambda_{\alpha} \left[ -\lambda (\tau-t)^{\nu} \right] \Omega(t) dt = \int_{a}^{\tau} (\tau-t)^{-\nu-a+1} E^{-1}_{-\nu,1,-a}(\tau-t)^{\nu-a+1} \Omega(t) dt
$$

(19)

where

$$
\Lambda_{\alpha} \left( -\lambda \tau^{\mu} \right) = \tau^{\nu-a+1} E^{-1}_{-\nu,1,-a}(\tau-t)^{\nu-a+1}
$$

with the Prabhakar function, denoted [1]:

$$
E_{\alpha,\beta}^{\gamma}(\tau) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\gamma+k)} \frac{\tau^k}{\Gamma(\alpha+k+1)}
$$

The right-sided general fractional-order integral of $\Omega(\tau)$ is defined:

$$
\left( \frac{b^{(\alpha)}}{b} \right)(\tau) = \int_{b}^{\tau} \Lambda_{\alpha} \left[ -\lambda (\tau-t)^{\nu} \right] \Omega(t) dt
$$

For $a = 0$, eq. (19) can be written:

$$
\left( \frac{b^{(\alpha)}}{b} \right)(\tau) = \int_{0}^{\tau} \Lambda_{\alpha} \left[ -\lambda (\tau-t)^{\nu} \right] \Omega(t) dt
$$

(21)

where $\Omega \in (a, b)$.

The Laplace transform of eq. (19) can be presented:

$$
L \left[ \left( \frac{b^{(\alpha)}}{b} \right)(\tau) \right] = s^{\nu-a+1} \left( 1 + \lambda s^{-a+1} \right) \Omega(s)
$$

(22)

A new application in the heat transfer process

In this section, a new general fractional-order derivative heat transfer model is presented.

We now consider the new general fractional-order derivative heat transfer model:

$$
\sigma \frac{d^{(\alpha)}}{dx^{(\alpha)}} X(x) = \chi
$$

(23)

with the initial value condition:

$$
X(x)|_{x=0} = X(0)
$$

(24)

where $\sigma$ represents the thermal conductivity of the material and $\chi$ – the heat flux density.

With the use of eq. (17), we have:

$$
\frac{1}{s^{\nu+1}} \frac{\sigma}{1 + \lambda s^{-(a+1)}} \left[ sX(s) - X(0) \right] = \chi
$$

which implies that:

$$
X(s) = \frac{\chi}{\sigma} (1 + \lambda s^{-a+1}) s^{\nu} + \frac{X(0)}{s}
$$

(26)

Finally, we have the solution of the general fractional-order derivative heat transfer model:

$$
X(x) = \frac{\chi}{\sigma} x^{-(a+1)} E^{-1}_{-\nu,1,-a} (-\lambda x^{\nu+1}) + X(0)
$$

(27)
Conclusion

In our work, we have addressed the new general fractional-order derivative of the Liouville-Caputo type without the singular kernel of the Rabotnov fractional-exponential function and its Laplace transform. As an potential application, the general fractional-order derivative heat transfer model and its solution based on the general Prabhakar function have been investigated in detail. The general fractional-order derivative is accurate and efficient for description of the general fractional-order dynamics in complex and power-law phenomena.

Acknowledgment

This research has been supported by the financial support of the 333 Project of Jiangsu Province, People’s Republic of China (Grant No. BRA2018320), the Yue-Qi Scholar of the China University of Mining and Technology (Grant No. 102504180004) and the State Key Research Development Program of the People’s Republic of China (Grant No. 2016YFC0600705).

Nomenclature

\( T(x) \) – temperature distribution, [K]  
\( x \) – space co-ordinate, [m]  
\( \mathcal{L}\{\cdot\} \) – Laplace transform, [–]  
\( \alpha \) – fractional order, [–]  
\( \kappa \) – thermal conductivity, [\( \text{W m}^{-1} \text{K}^{-1} \)]  
\( \chi \) – heat flux density, [\( \text{W m}^{-2} \)]

References