A NEW GENERAL FRACTIONAL-ORDER WAVE MODEL INVOLVING MILLER-ROSS KERNEL

by

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In the paper we consider a general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel for the first time. The analytical solution for the general fractional-order wave model is investigated in detail. The obtained result is given to explore the complex processes in the mining rock.

Key words: fractional-order wave model, general fractional-order derivative, Miller-Ross kernel, mining rock

Introduction

The mathematical model for the wave propagation in the mining rock has been investigated by many scientists, see [1-4] and references cited therein. For example, the linear model for the wave propagation:

\[
\frac{\partial^2 \mathcal{R}(x,t)}{\partial t^2} = \phi_1 \frac{\partial}{\partial x} \left[ \phi_1 \frac{\partial \mathcal{R}(x,t)}{\partial x} \right]
\]

(1)

where \( \phi_1 \) is a constant and \( \mathcal{R}(x, t) \) – the wave function, was proposed in [5]. As a special case of (1), the linear model for the wave propagation:

\[
\frac{\partial^2 \mathcal{R}(x,t)}{\partial t^2} = \phi_2 \frac{\partial^2 \mathcal{R}(x,t)}{\partial x^2}
\]

(2)

where \( \phi_2 \) is a constant and \( \mathcal{R}(x, t) \) is the wave function, was proposed in [6]. The models can be used to describe 1-D wave propagation in the mining rock. Recently, a general fractional-order derivative within the Miller-Ross kernel [7]. The main aim of the article is to propose the general fractional-order derivative model for the wave propagation based on the general fractional-order derivative involving the Miller-Ross kernel [8] and to investigate its analytical solution.

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A general fractional-order calculus involving the Miller-Ross kernel

In this section, we introduce the general fractional-order derivative involving the special function, which proposed in by Miller and Ross, see [8], from the point of view of the general fractional-order calculus application.

The Miller-Ross function and its Laplace transform

For the given real constant, $\lambda$, the Miller-Ross function with one-parameter constant $\lambda$ is defined [7, 8]:

$$
\varphi_\alpha(\lambda t) = \Gamma^\alpha \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma(k+\alpha)} = \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma(k+1+\alpha)}
$$

(1)

with the Laplace transform [8]:

$$
L\{\varphi_\alpha(\lambda t)\} = L\left(\sum_{k=0}^{\infty} \frac{\lambda^k t^k}{\Gamma(k+1+\alpha)}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{s^{k+\alpha+1}} = s^{-(\alpha+1)} \left(1 - \lambda s^{-1}\right)^{-\alpha} \left(\lambda s^{-1} < 1\right)
$$

(2)

where the Laplace transform operator of the function $u(t)$ is represented [7]:

$$
\mathcal{L}[u(t)] = U(s) = \int_0^\infty e^{-st} u(t) \, dt
$$

(3)

A general fractional-order integral operators involving the Miller-Ross kernel

The left-sided general fractional-order integral operator involving the Miller-Ross kernel is defined [7]:

$$
\text{MR} I^\alpha_\alpha f(t) = \int_a^t \varphi_\alpha \left[-\lambda (t-\tau)^\alpha\right] j(\tau) \, d\tau
$$

(4)

and the right-sided general fractional-order integral operator involving the Miller-Ross kernel:

$$
\text{MR} I^\alpha_\beta f(t) = \int_\beta^t \varphi_\alpha \left[-\lambda (\tau-t)^\alpha\right] j(\tau) \, d\tau
$$

(5)

When $a = 0$, the general fractional-order integral operator involving the Miller-Ross kernel become:

$$
\text{MR} I^\alpha_0 f(t) = \int_0^t \varphi_\alpha \left[-\lambda (t-\tau)^\alpha\right] j(\tau) \, d\tau
$$

(6)

with its Laplace transform:

$$
\mathcal{L}\left[\text{MR} I^\alpha_0 f(t)\right] = s^{\alpha+1} \left(1 - \lambda s^{-1}\right)^{-\alpha} f(s)
$$

(7)

General fractional-order derivatives involving the Miller-Ross kernel

The left-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel is defined [7]:

$$
\text{MR} D^\alpha_\alpha f(t) = \frac{d^\alpha}{dt^\alpha} \left[\text{MR} I^{\alpha-\alpha}_\alpha f(t)\right] = \frac{d^\alpha}{dt^\alpha} \int_a^t \varphi_\alpha \left[-\lambda (t-\tau)^\alpha\right] j(\tau) \, d\tau
$$

(8)
and the right-sided general fractional-order derivative of the Riemann-Liouville type involving the Miller-Ross kernel:

\[
D_{a^+,b}^{\alpha,\lambda} j(t) = \left[ -\frac{d}{d\tau} \right]_a^b j(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j(\tau) \, d\tau
\]

(9)

where \( \alpha \) is the positive integer numbers.

The left-sided general fractional-order derivative of the Liouville-Sonine type involving the Miller-Ross kernel is defined \([7]\):

\[
D_{a-}^{\alpha,\lambda} j(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j^{(1)}(\tau) \, d\tau
\]

(10)

and the right-sided general fractional-order derivative of the Liouville-Sonine type within the Miller-Ross kernel:

\[
D_{b+}^{\alpha,\lambda} j(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j^{(1)}(\tau) \, d\tau
\]

(11)

The left-sided general fractional-order derivative of the Liouville-Sonine-Caputo type involving the Miller-Ross kernel is defined \([7]\):

\[
D_{a-}^{\alpha,\lambda} j(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j^{(1)}(\tau) \, d\tau
\]

(12)

and the right-sided general fractional-order derivative of the Liouville-Sonine-Caputo type within the Miller-Ross kernel:

\[
D_{b+}^{\alpha,\lambda} j(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j^{(1)}(\tau) \, d\tau
\]

(13)

The relation between the general fractional-order derivative of the Riemann-Liouville and Liouville-Sonine types is given \([7]\):

\[
D_{a-}^{\alpha,\lambda} j(t) = D_{a-}^{\alpha,\lambda} j(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \phi_\alpha \left[ -\lambda (\tau - t)^\alpha \right] j(\tau) \, d\tau
\]

(14)

The Laplace transforms of the general fractional-order derivatives can be given \([7]\):

\[
\mathcal{L}_{s} \left[ D_{a-}^{\alpha,\lambda} j(t) \right] = s^{-\alpha - 1} (1 + \lambda s^{-1})^{-1} j(s)
\]

(15)

and

\[
\mathcal{L}_{s} \left[ D_{b+}^{\alpha,\lambda} j(t) \right] = s^{-\alpha - 1} (1 + \lambda s^{-1})^{-1} \left[ j(s) - j(0) \right]
\]

(16)

A new general fractional-order wave model

We now consider a new general fractional-order wave model containing the general fractional-order derivative of the Liouville-Sonine type within the Miller-Ross kernel:

\[
D_{a-}^{\alpha,\lambda} j(t) = \frac{\partial^\beta R(x,t)}{\partial x^\beta} \quad (x > 0, \ t > 0)
\]

(18)

subjected to the initial and boundary conditions:

\[
R(t) (x, 0) = 0, \ R(x, 0) = 0, \ R(0,t) = 0, \ R(+\infty, t) = 0
\]

(19)
where the general fractional-order partial derivatives of orders 2 and 1 are defined:

\[
\left( \frac{\partial^{2 \lambda}}{\partial t^{2 \lambda}} \right) \mathcal{R}(x,t) = \int_0^t \phi_{\alpha} \left[ -\lambda (t-\tau)^\alpha \right] \mathcal{R}^{(2)}(x,\tau) \, d\tau
\]

and

\[
\left( \frac{\partial^{1 \lambda}}{\partial t^{1 \lambda}} \right) \mathcal{R}(x,t) = \int_0^t \phi_{\alpha} \left[ -\lambda (t-\tau)^\alpha \right] \mathcal{R}^{(1)}(x,\tau) \, d\tau
\]

respectively.

With the use of the Laplace transform, we present:

\[
\frac{\partial^2 \mathcal{R}(x,s)}{\partial x^2} = s^{-\alpha \lambda} \left( 1 + \lambda s^{-1} \right)^{-1} \mathcal{R}(x,s)
\]

with the general solution, given:

\[
\mathcal{R}(x,s) = \Lambda_1 e^{-\sqrt{s^{-\alpha \lambda}} (1+\lambda s^{-1})^{-1/2}} + \Lambda_2 e^{\sqrt{s^{-\alpha \lambda}} (1+\lambda s^{-1})^{-1/2}}
\]

where \( \Lambda_1 \) and \( \Lambda_2 \) are the constants.

Finally, we have \( \Lambda_2 = 0 \) and \( \Lambda_1 = 0 \) and such that:

\[
\mathcal{R}(x,s) = e^{-\sqrt{s^{-\alpha \lambda}} (1+\lambda s^{-1})^{-1/2}}
\]

Thus, we have:

\[
\mathcal{R}(x,t) = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+1)} \frac{(a-1)n}{2} \left( 1 + \lambda s^{-1} \right)^{-n/2}
\]

which leads to:

\[
\mathcal{R}(x,t) = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+1)} \frac{(a-1)n}{2} E_{\frac{a-1}{2}} \left( -\lambda t \right)
\]

where the Laplace transform of the generalized Prabhakar function is written [7]:

\[
3 \left( t^{-\frac{(a-1)n}{2}} E_{\frac{a-1}{2}} \left( -\lambda t \right) \right) = \sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma(n+1)} \frac{(a-1)n}{2} \left( 1 + \lambda s^{-1} \right)^{-n/2}
\]

Conclusion

In our task, we investigate the new general fractional-order wave model with the general fractional-order derivative involving the Miller-Ross kernel. With the aid of the Laplace transform, we obtain the analytical solution. The special functions are accurate and efficient for descriptions of the mining rock.

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Nomenclature

\( \alpha \) – fractional order, [-]
\( t \) – time co-ordinate, [m]
\( x \) – space co-ordinate, [m]
References