THE GENERALIZED GIACHETTI-JOHNSON HIERARCHY AND ALGEBRO-GEOMETRIC SOLUTIONS OF THE COUPLED KdV-MKdV EQUATION

by

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By using a Lie algebra A1, an isospectral Lax pair is introduced from which a generalized Giachetti-Johnson hierarchy is generated, which reduce to the coupled KdV-MKdV equation, furthermore, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed in terms of Riemann theta functions.

Key words: algebro-geometric solution, Riemann theta function, coupled KdV-MKdV equation

Introduction

In recent years, a family of methods were developed to find the exact solutions for the linear and non-linear PDE. Among them, there are adomian decomposition method [1], traveling wave transformation method [2, 3], Riccati equation method [4] and algebro-geometric method [5-7]. The mathematical model of shallow water waves was rediscovered by Korteweg and de Vries [8], which is commonly known as KdV equation, many physical quantities of KdV equation and MKdV equation have been discussed later [9-11]. In this paper, we first use a Lie algebra A1 to obtain the generalized Giachetti-Johnson (GGJ) hierarchy, which reduce to the coupled KdV-MKdV equation, Then in terms of Riemann theta functions, the algebro-geometric solutions of the coupled KdV-MKdV equation are constructed.

The GGJ hierarchy and coupled KdV-MKdV equation

The Lie algebra A1 has a basis [12, 13]:

\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_1, \quad [e_2, e_3] = e_3 \] (1)

Consider the isospectral problem:

\[ \psi_x = U \psi, \quad U = (\alpha \lambda + s) e_1 (0) + u e_2 (0) + (\alpha + u) e_3 (0), \]
\[ \psi_t = V \psi, \quad V = \sum_{\text{m>0}} \left[ a_m e_1 (-m) + b_m e_2 (-m) + c_m e_3 (-m) \right] \] (2)

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Note:

\[ V_+^{(n)} = \sum_{n=0}^{\infty} \left[ a_n e_1 (n-m) + b_n e_2 (n-m) + c_n e_3 (n-m) \right] \]

\[ -V_+^{(n)} + \left[ U, V_+^{(n)} \right] = -2a b e_2 (0) + 2a c e_3 (0) \]

Set \( V^{(n)} = V^{(n)} - a_n e_1 (0) \), then \( U - V^{(n)} + \left[ U, V^{(n)} \right] \) leads to the GGJ hierarchy:

\[ u_n = \begin{pmatrix} u_1 \\ u_2 \\ s \end{pmatrix} = \begin{pmatrix} 0 & \partial - 2s & 0 \\ \partial + 2s & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \partial \end{pmatrix} \begin{pmatrix} c_n \\ b_n \\ 2a_n \end{pmatrix} \]

(3)

When taking \( n = 3, s = 0 \) in eq. (3), we have the coupled KdV-MKdV equation:

\[ u_{t_0} = \frac{1}{4} \alpha^{-3} [u_{xxx} - 4u_t (\alpha_1 + u_2) u_{xx} - 2u_1^2 u_{xx} , u_{t_0} = \]

\[ = \frac{1}{4} \alpha^{-3} [u_{xxx} - 4u_t (\alpha_1 + u_2) u_{xx} - 2(\alpha_1 + u_2)^2 u_{x} ] \]

(4)

Algebro-geometric solutions of the coupled KdV-MKdV equation

We introduce the Lenard gradient sequence \( \{ S_j \}_{j=0}^{\infty} \):

\[ K_j S_j = J S_{j+1}, S_0 = (0,0,1)^T \]

(5)

\[ K = \begin{pmatrix} 0 & \partial & 2u_1 \\ -\partial & 0 & 2(\alpha_1 + u_2) \\ -u_1 & \alpha_1 + u_2 & \partial \end{pmatrix}, J = \begin{pmatrix} 0 & 2\alpha & 0 \\ 2\alpha & 0 & 0 \\ -u_1 & \alpha_1 + u_2 & \partial \end{pmatrix} \]

Let \( X = (X_1, X_2)^T \) and \( Y = (Y_1, Y_2)^T \) be two basic solutions of spectral problems:

\[ \Phi_s = U \Phi, U = \begin{pmatrix} \alpha \lambda & u_1 \\ \alpha_1 + u_2 & -\alpha \lambda \end{pmatrix}, \quad \Phi_{s_0} = V^{(m)} \Phi, \quad V^{(m)} = \begin{pmatrix} A^{(m)} & B^{(m)} \\ C^{(m)} & -A^{(m)} \end{pmatrix} \]

\[ A^{(m)} = -a_m + \sum_{j=0}^{\infty} a_j^m \lambda^{n-j}, \quad B^{(m)} = \sum_{j=0}^{\infty} b_j^m \lambda^{n-j}, \quad C^{(m)} = \sum_{j=0}^{\infty} c_j^m \lambda^{n-j} \]

(7)

then:

\[ W = \frac{1}{2} (XY^T + YY^T) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \hat{g} & \hat{f} \\ \hat{h} & -\hat{g} \end{pmatrix} \]

satisfies the Lax equation:

\[ W_s = [U, W], \quad W_{t_n} = [V^{(n)}, W] \]

(8)

which implies that \( \det W \) is a constant independent of \( x \) and \( t_n \). From eq. (8), we get:

\[ \hat{g}_s = u_1 \hat{h} - (\alpha_1 + u_2) \hat{f}, \quad \hat{f}_s = 2\alpha \hat{f} - 2u_1 \hat{g}, \quad \hat{h}_s = -2\alpha \lambda \hat{h} + 2(\alpha_1 + u_2) \hat{g} \]

\[ \hat{g}_{t_n} = B^{(n)} \hat{h} - C^{(n)} \hat{f}, \quad \hat{f}_{t_n} = 2A^{(n)} \hat{f} - 2B^{(n)} \hat{g}, \quad \hat{h}_{t_n} = 2C^{(n)} \hat{g} - 2A^{(n)} \hat{h} \]
\[ \hat{g} = \sum_{j=0}^{N+1} \hat{g}_j \lambda^{N+1-j}, \quad \hat{f} = \sum_{j=0}^{N+1} \hat{f}_j \lambda^{N+1-j}, \quad \hat{h} = \sum_{j=0}^{N+1} \hat{h}_j \lambda^{N+1-j}, \]

\[ KQ_{j-1} = JQ_j, \quad JQ_0 = 0, KQ_{N+1} = 1 \]

\[ Q_j = (\hat{h}, \hat{f}, \hat{g})^T, \quad Q_0 = \beta_0 S_0 = \beta_0 (0,0,1)^T, \quad Q_1 = \sum_{j=0}^{N} \beta_j S_{j-1}, k = 0,1, \ldots, \beta_0 S_N + \ldots + \beta_N S_N = 0 \quad (12) \]

Set \( \beta_0 = 1 \) in eq. (12), from eqs. (5) and (12), we have:

\[ Q_1 = \left( \frac{\alpha^{-1}(\alpha_i + u_i)}{\beta_i}, \frac{\alpha^{-1}u_i}{\alpha_i + u_i}, \frac{-\frac{1}{2} \alpha^{-2} u_i + \alpha^{-2} \beta_i (\alpha_i + u_i)}{\beta_i} \right), \quad Q_2 = \left( \frac{\alpha^{-2} u_i}{\alpha_i + u_i}, \frac{-\frac{1}{2} \alpha^{-2} u_i + \alpha^{-2} \beta_i (\alpha_i + u_i)}{\beta_i}, \frac{-\frac{1}{2} \alpha^{-2} u_i}{\alpha_i + u_i} + \beta_2 \right) \]

\[ \hat{f} = \alpha^{-1} u_i \prod_{j=1}^{N} (\lambda - \mu_j), \quad \hat{h} = \alpha^{-1} (\alpha_i + u_i) \prod_{j=1}^{N} (\lambda - \nu_j) \quad (13) \]

By comparing the coefficients of \( \lambda^{N-1}, \lambda^{N-2} \) and combining eqs. (11) and (13), we have:

\[ \frac{1}{2} \alpha^{-1} u_i + \beta_1 = -\sum_{j=1}^{N} \mu_j, \quad -\frac{1}{2} \alpha^{-1} (\alpha_i + u_i) + \beta_1 = -\sum_{j=1}^{N} \nu_j \quad (14) \]

\[ \frac{1}{4} \alpha^{-3} \left[ \frac{u_i}{u_i - 2u_i (\alpha_i + u_i)} - 2u_i (\alpha_i + u_i) \right] + \frac{1}{2} \alpha^{-3} \beta_i \frac{u_i}{u_i} + \beta_2 = \sum_{j=1}^{N} \mu_j \mu_i \quad (15) \]

\[ -\frac{1}{4} \alpha^{-3} \left[ \frac{(\alpha_i + u_i)}{\alpha_i + u_i} - 2u_i (\alpha_i + u_i) \right] - \frac{1}{2} \alpha^{-3} \beta_i \frac{u_i}{\alpha_i + u_i} + \beta_2 = \sum_{j=1}^{N} \nu_j \nu_i \quad (16) \]

\[ -\det W = \hat{g}^2 + \hat{f} \hat{h} = \prod_{j=1}^{2N+2} (\lambda - \lambda_j) = R(\lambda) \quad (17) \]

\[ 2 \hat{g}_0 \hat{g}_1 = -\sum_{j=1}^{2N+2} \lambda_j, \quad \hat{g}_1^2 = 2 \hat{g}_0 \hat{g}_2 + \hat{f} \hat{h}_1 = \sum_{j=1}^{N} \lambda_j \lambda_3 \]

\[ \beta_1 = -\frac{1}{2} \sum_{j=1}^{N} \lambda_j, \quad \beta_3 = \frac{1}{2} \left[ \sum_{j=1}^{N} \lambda_j \lambda_3 - \frac{1}{4} \left( \sum_{j=1}^{2N+2} \lambda_j \right)^2 \right] \quad (18) \]

Thus, we get:

\[ \hat{g} \mid_{\lambda=\mu_1} = \sqrt{R(\mu_1)}, \quad \hat{f} \mid_{\lambda=\mu_1} = -\alpha^{-1} u_i u_i \prod_{j=1}^{N} (\mu_k - \mu_j) = -2u_i \hat{g} \mid_{\lambda=\mu_1}, \quad \mu_1 = \frac{2 \alpha \sqrt{R(\mu_1)}}{\prod_{j=1}^{N} (\mu_k - \mu_j)} \]

\[ \hat{g} \mid_{\lambda=\nu_1} = \sqrt{R(\nu_1)}, \quad \hat{h} \mid_{\lambda=\nu_1} = -\alpha^{-1} (\alpha_i + u_i) v_i \prod_{j=1}^{N} (v_k - v_j) = 2(\alpha_i + u_i) \hat{g} \mid_{\lambda=\nu_1} \quad (19) \]

\[ \nu_1 = \frac{2 \alpha \sqrt{R(\nu_1)}}{\prod_{j=1}^{N} (v_k - v_j)} \]
which gives rise to:

\[
\mu_{ki} = \frac{2\alpha \sqrt{R(\mu_i)}}{\prod_{j=1, j \neq k}^N (\mu_i - \mu_j)}, \quad v_{ki} = \frac{2\alpha \sqrt{R(v_i)}}{\prod_{j=1, j \neq k}^N (v_i - v_j)}
\]  

(20)

\[
\mu_k = 2 \sum_{j=1}^N \left( \sum_{j=1}^N \mu_j + \beta_k \right) \mu_k + \sum_{j=1}^N u_j \mu_k + \left( \sum_{j=1}^N \mu_j + \beta_k \right) \beta_k - \beta_k \right] \sqrt{R(\mu_k)}
\]

\[
\prod_{j=1, j \neq k}^N (\mu_k - \mu_j)
\]

\[
v_k = \frac{-2 \left( \sum_{j=1}^N \left( \sum_{j=1}^N \beta_j \right) v_k + \sum_{j=1}^N v_j \right) - \sum_{j=1}^N \left( \sum_{j=1}^N \beta_j \right) \beta_k - \beta_k \right] \sqrt{R(v_k)}
\]

(21)

\[
\prod_{j=1, j \neq k}^N (v_k - v_j)
\]

then \((u_1, u_2)\) determined by eq. (14) is a solution of eq. (4).

We consider the hyper-elliptic Riemann surface:

\[
\Gamma \colon \xi^2 = R(\lambda), \quad R(\lambda) = \prod_{k=1}^{2N+2} (\lambda - \lambda_k), \quad \lambda_{2N+2} = 0
\]

for a fixed point \(p_0\), we introduce the Abel-Jacobi co-ordinate:

\[
\rho_m = (\rho_m^{(1)}, \rho_m^{(2)}, \ldots, \rho_m^{(N)}), \quad m = 1, 2
\]

with

\[
\rho^{(j)}(x, t) = \sum_{k=1}^N \int_{p_0}^{\rho_s} d\lambda \frac{\lambda^{j-1} d\lambda}{\sqrt{R(\lambda)}}, \quad \rho^{(j)}(x, t) = \sum_{k=1}^N \int_{p_0}^{\rho_s} d\lambda \frac{\lambda^{j-1} d\lambda}{\sqrt{R(\lambda)}}
\]

\[
\delta_{ij}, i = 1, 2, \ldots, N
\]

(22)

In a similar way, we obtain from (20)-(22):

\[
\bar{\partial}_{\rho^{(j)}_1} = \Omega^{(j)}_1 = 2(C_{j, N+2} - \beta_j C_{j, N+1} + \beta_j^2 C_{j, N+1} - \beta_j C_{j, N+1}), \quad \bar{\partial}_{\rho^{(j)}_2} = -\Omega^{(j)}_1, \quad \bar{\partial}_{\rho^{(j)}_2} = -\Omega^{(j)}_1
\]

\[
\rho_2 = -\Omega_0 x - \Omega t + \gamma_2, \quad \gamma^{(j)}_1 = \sum_{k=1}^N \int_{p_0}^{\rho_s} d\lambda \frac{\lambda^{j-1} d\lambda}{\sqrt{R(\lambda)}}, \quad \gamma^{(j)}_2 = \sum_{k=1}^N \int_{p_0}^{\rho_s} d\lambda \frac{\lambda^{j-1} d\lambda}{\sqrt{R(\lambda)}}
\]

\[
\gamma_n = (\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_N^{(1)}), \quad m = 1, 2
\]

We define an Abel map on \(\Gamma\):

\[
A(p) = \int_{p_0}^{\rho_s} d\lambda \omega = (\omega_1, \ldots, \omega_N)^t, \quad A \left( \sum n_k p_k \right) = \sum n_k A(p_k)
\]

Consider two special divisors \(\sum_{k=1}^N P^{(k)}_m\) \((m = 1, 2)\), then we have:
with
\[ p_1^{(k)} = [\mu_k, \xi(\mu_k)] \] and \[ p_2^{(k)} = [\nu_k, \xi(\nu_k)] \]

The Riemann theta function of \( \Gamma \) is defined:
\[ \theta(z, \zeta) = \sum \exp(\pi i (\tau z + \zeta)), \quad \zeta \in \mathbb{C}^N \]

where
\[ \zeta = (\zeta_1, \ldots, \zeta_N)^T, \quad (\zeta, z) = \sum_{k=1}^N \zeta_k z_k \]

Then we have:
\[ \sum_{j=1}^N \mu_j = I - \frac{1}{2} \sum_{j=1}^N \text{Re} s_{j-\nu_j}, \lambda \text{d} \ln F_1(\lambda), \quad \sum_{j=1}^N \nu_j = I - \frac{1}{2} \sum_{j=1}^N \text{Re} s_{j-\mu_j}, \lambda \text{d} \ln F_2(\lambda) \]
\[ F_w(z^{-1}) = \theta_s^{(m)} + z(-1)^{s-1} \sum_{j=1}^N c_j \theta_j^{(m)} + o\left(z^2\right) \quad (23) \]

where
\[ \theta_s^{(m)} = \theta(\rho_m + M_m + \eta), \theta(\cdots, \rho_m^{(j)} + M_m^{(j)} + \eta^{(j)}, \cdots) \]

It is easy to calculate that:
\[ \partial_s \theta_s^{(m)} = \sum_{j=1}^N 2 \alpha C_{jN} D_j \theta_s^{(m)} \quad (24) \]

Substituting eq. (24) into eq. (23), we have:
\[ \frac{d}{dz} \ln F_w(z^{-1}) = \frac{1}{2} \alpha^{-1}(-1)^{s-1} \partial_s \ln \theta_s^{(m)} + o\left(z\right) \quad F_w(z^{-1}) = \theta_s^{(m)} + \frac{1}{2} \alpha^{-1}(-1)^{s-1} \partial_s \theta_s^{(m)} + o\left(z^2\right) \quad (25) \]

where
\[ \theta_s^{(1)} = \theta(\Omega_x + \Omega t + \pi), \quad \theta_s^{(2)} = \theta(-\Omega_x - \Omega t + \eta) \]

From eq. (23), we have:
\[ \sum_{j=1}^N \mu_j = I + \frac{1}{2} \alpha^{-1} \partial_s \ln \theta_s^{(1)} + \frac{1}{2} \sum_{j=1}^N \nu_j = I + \frac{1}{2} \alpha^{-1} \partial_s \ln \theta_s^{(2)} \quad (26) \]

Substituting eq. (26) into eq. (14), we obtain the algebro-geometric solutions of the coupled KdV-MKdV eq. (4):
\[ u_1 = \frac{\theta_s^{(1)}}{\theta_s^{(1)}} \exp \left[ \alpha \left( 2I - \sum_{j=1}^N \lambda_j \right) x + u_1^i(t) \right], \quad u_2 = \frac{\theta_s^{(2)}}{\theta_s^{(2)}} \exp \left[ -\alpha \left( 2I - \sum_{j=1}^N \lambda_j \right) x + u_2^i(t) \right] - \alpha_1 \]
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Nomenclature

$t$ – time, [s]

$x$ – space, [m]

References


