On the heat transfer to an accelerated translating droplet

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Abstract
An analysis is made for the transient response behavior of the both, outer and inner, thermal boundary layers of a fluid sphere moving at constant acceleration with internal circulation in another viscous fluid of large extent initially at rest under the condition of large Reynolds and Peclet numbers. The disturbance is initiated by a step change in temperature of either the continuous or disperse region fluids. The approximate solutions of the governing energy equations are found by using the inviscid approximations for the flow fields.

Keywords: thermal boundary layer, interface, accelerated spherical droplet, average Nusselt number

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Nomenclature \(^1\)

\(T\) temperature
\(R\) droplet radius
\(\ddot{z}\) droplet translatory acceleration
\(t\) time
\(\tau\) dimensionless time
\(r\) spherical radial coordinate
\(\theta\) spherical polar angle measured from front stagnation
\(q\) heat flux
\(N_u\) Nusselt number
\(a\) thermal diffusivity
\(\lambda\) thermal conductivity
\(\mu\) viscosity
\(\nu\) kinematic viscosity
\(\rho\) density

Subscripts

\(0\) refers to initial condition
\(e\) refers to outer region
\(i\) refers to inner region
\(\infty\) refers to large distance from the droplet

1 Introduction

A knowledge of the heat transfer associated with a moving droplet is of importance to a variety of industrial processes. So, in a number of direct contact exchangers, for instance, one fluid is dispersed in the form of droplets moving in another fluid. The monograph by Levich [1] gave an account of the methods of analysis and the problems related to the prediction of diffusional flux to a moving drop.

Boussinesque [2] was the first to obtain a solution for the heat transfer rate from a fluid sphere of constant surface temperature, moving at

\(^1\)All quantities with an overscore denote averages over the droplet surface.
a constant speed in another fluid of infinite extent. Major assumptions used in the analysis were: constant properties, irrotational flow field, and thin thermal layer. A number of theoretical analyses of the steady motion of a gas bubble in liquids at large Reynolds numbers have been published since 1949. The first solutions were due to Levich [3] who assumed the bubble caused a small perturbation in the basic inviscid flow. Levich’s solution has been improved by Chao [4] and Moore [5] through the reexamination of the linearized steady-state boundary layer equations. The linearizing theory has also been extended to the steady motion of a liquid drop in another liquid of comparable density and viscosity by Harper and Moore [6] and by Parlange [7].

The process of unsteady boundary-layer formation was examined especially past a body impulsively started, much more rarely for the accelerated motions. So, the boundary-layer formation in flow past a fluid sphere, assuming that the internal motion of the enclosed gas has a negligible effect on the external liquid motion, was studied by Chen [8], describing the boundary-layer growth on a spherical bubble due to an initial discontinuity in tangential stress at the bubble surface. The both outer and inner flows of a spherical gas bubble started impulsively from rest in a viscous liquid of infinite extent was examined in [12]. Finally, the transient heat and mass transfer to a fluid sphere moving at constant velocity in another fluid was studied by Chao [9]. The governing energy or mass concentration equations are solved using similarity transformations in the case of the inviscid external and internal flow fields. This paper deals with the both outer and inner thermal boundary layers due to a spherical droplet put into the accelerated motion from rest.

2 Analysis

Let us consider a fluid sphere of radius $R$ put into the motion with constant acceleration $\ddot{z}$ in another fluid of infinite extent. We imagine that at time $t=0$ the temperature of the continuous phase fluid undergoes a step change from an initial uniform and constant temperature $T_0$ to $T_\infty$. It is desired to examine the transient response behaviour of the thermal boundary layers both inside and outside of the droplet.
We assume a fully developed internal circulation. Winnikov and Chao [11] have experimentally demonstrated that, in highly purified systems, moving droplets invariably exhibit internal circulation. On the other hand, since only large droplet Reynolds number is of interest, we are going to use the inviscid approximations for the flow fields. This is particularly true if the internal circulation is vigorous. Accordingly, the external flow is irrotational and the internal field is that of Hill’s spherical vortex. Generally speaking, the viscous effect is small when the Reynolds number exceeds two or three hundred. It may be of interest to note that if the hydrodynamic boundary layers are developing simultaneously with the thermal boundary layer, the inviscid approximation is even better[10]. Under the condition of large Peclet number, the thermal boundary layers are thin except the region close to the rear stagnation. Finally, we accept also two usual assumptions of constant fluid properties and negligible dissipation.

The coordinate system and velocity components are shown in Fig.1. All quantities with subscript “e” refer to the continuous region fluid while subscript “i” refers to the disperse region. We note that $y$ is positive in the external flow and negative in the internal flow.

The inviscid flow fields are well known: the radial and circumferential velocity components are:

$$v_{re} = -U_\infty (1 - R^3/r^3) \cos \theta,$$

$$v_{\theta e} = U_\infty (1 + \frac{1}{2} R^3/r^3) \sin \theta$$

for the external flow, and

$$v_{ri} = \frac{3}{2} U_\infty (1 - r^2/R^2) \cos \theta,$$

$$v_{\theta i} = -\frac{3}{2} U_\infty (1 - 2r^2/R^2) \sin \theta$$

for the internal flow. The latter was first given by Hill. We now replace the radial coordinate $r$ by $y$ which has its origin at the drop surface. Thus

$$y = r - R \quad \text{and} \quad |y|/R << 1.$$
Within the context of thin boundary layers, the following approximations for the velocity components are valid:

\[ v_{\theta e} = v_{\theta i} = \frac{3}{2} \ddot{z} t \sin \theta, \]  

(1)

\[ v_{re} = -v_{ri} = -3\ddot{z} t \frac{|y|}{R} \cos \theta, \]  

(2)

since

\[ \begin{aligned}
R^3/r^3 &= (1 + y/R)^{-3} = 1 - 3 \frac{y}{R} \\
&= 1 - \frac{3}{R} y
\end{aligned} \]

\[ \begin{aligned}
r^2/R^2 &= (1 + y/R)^2 = 1 + 2 \frac{y}{R} \\
&= 1 + \frac{2}{R} y
\end{aligned} \]

when \( \frac{y}{R} \ll 1 \)

and for the case of uniform acceleration of the droplet:

\[ \dot{z} = \frac{d\dot{z}}{dt} = \text{const} \quad \text{whence} \quad \dot{z} \equiv U_{\infty} = \ddot{z} t. \]
Following the usual procedure of making order of magnitude estimates of the various terms in the governing conservation equations, we establish that the energy equations for the thermal boundary layers are:

\[
\frac{\partial T_e}{\partial t} - 3 \ddot{z} t \cos \theta \frac{y}{R} \frac{\partial T_e}{\partial y} + \frac{3}{2} \ddot{z} t \sin \theta \frac{1}{R} \frac{\partial T_e}{\partial \theta} = a_e \frac{\partial^2 T_e}{\partial y^2}, \quad y > 0 \tag{3}
\]

\[
\frac{\partial T_i}{\partial t} - 3 \ddot{z} t \cos \theta \frac{y}{R} \frac{\partial T_i}{\partial y} + \frac{3}{2} \ddot{z} t \sin \theta \frac{1}{R} \frac{\partial T_i}{\partial \theta} = a_i \frac{\partial^2 T_i}{\partial y^2}, \quad y < 0. \tag{4}
\]

The initial and boundary conditions are:

\((y > 0)\)

\[T_e(y, \theta, 0) = T_\infty \tag{5e}\]
\[T_e(\infty, \theta, t) = T_\infty \tag{6e}\]
\[\frac{\partial T_e}{\partial \theta}(y, 0, t) = \frac{\partial T_i}{\partial \theta}(y, \pi, t) = 0 \tag{7e}\]

\((y < 0)\)

\[T_i(y, \theta, 0) = T_0 \tag{5i}\]
\[T_i(-\infty, \theta, t) = T_0 \tag{6i}\]
\[\frac{\partial T_i}{\partial \theta}(y, 0, t) = \frac{\partial T_i}{\partial \theta}(y, \pi, t) = 0 \tag{7i}\]

and, at the interface:

\[T_e(0, \theta, t) = T_i(0, \theta, t) \tag{8}\]
\[\lambda_e \frac{\partial T_e}{\partial y}(0, \theta, t) = \lambda_i \frac{\partial T_i}{\partial y}(0, \theta, t). \tag{9}\]

Conditions \((7e)\) and \((7i)\) follow from the requirement of symmetry at \(\theta = 0\) and \(\theta = \pi\), and condition \((6i)\) should strictly be written as

\[T_i(-\delta_{th}, \theta, t) = T_0,\]
where $\delta_{th}$ is the thermal boundary-layer thickness. However, due to
the parabolic nature of the governing equations, it is permissible to
use (6i).

The integration of the unsteady thermal boundary layer equations
(3) and (4) can be carried out in most cases by a process of succes-
sive approximations, the method being based on the following physical
reasoning: In the first instant, after the motion had started from rest,
the thermal boundary layers are very thin and the terms $a_e \partial^2 T_e / \partial y^2$
and $a_i \partial^2 T_i / \partial y^2$ in equations (3) and (4) are very large, whereas the
convective terms retain their normal values. These two terms are then
balanced by the unsteady temperature variations $\partial T_e / \partial t$ and $\partial T_i / \partial t$.
Selecting a system of coordinates, as it is shown in Fig.1, which is at
rest with respect to the body, we consider a stationary fluid sphere sit-
uated in an upflowing unbound fluid (instead of a falling droplet) and
we can make the assumption that the both temperatures are composed
of two terms as follows:

$$T_e = T_{e1} + T_{e2}, \quad (10e)$$

$$T_i = T_{i1} + T_{i2}. \quad (10i)$$

Under these conditions the first approximations $(T_{e1}, T_{i1})$ satisfy the
following differential equations:

$$\frac{\partial T_{e1}}{\partial t} - a_e \frac{\partial^2 T_{e1}}{\partial y^2} = 0 \quad (11e)$$

$$\frac{\partial T_{i1}}{\partial t} - a_i \frac{\partial^2 T_{i1}}{\partial y^2} = 0, \quad (11i)$$

with the boundary conditions (5) to (9). The equations for the second
approximations $(T_{e2}, T_{i2})$ are obtained with reference to equations (3)
and (4) in which the convective terms are calculated from $T_{e1}$ and $T_{i1}$
previously found. Hence we have

$$\frac{\partial T_{e2}}{\partial t} - a_e \frac{\partial^2 T_{e2}}{\partial y^2} = 3 \ddot{z} t \cos \theta \frac{y}{R} \frac{\partial T_{e1}}{\partial y} - \frac{3}{2} \ddot{z} t \sin \theta \frac{1}{R} \frac{\partial T_{e1}}{\partial \theta}, \quad (12e)$$

$$\frac{\partial T_{i2}}{\partial t} - a_i \frac{\partial^2 T_{i2}}{\partial y^2} = 3 \ddot{z} t \cos \theta \frac{y}{R} \frac{\partial T_{i1}}{\partial y} - \frac{3}{2} \ddot{z} t \sin \theta \frac{1}{R} \frac{\partial T_{i1}}{\partial \theta}. \quad (12i)$$
The solutions of (11e) and (11i), satisfying all the conditions (5) to (9), can be obtained in the form:

\[ T_{e1} = T_{\infty} + \frac{T_0 - T_{\infty}}{1 + \beta} \text{erfc}\eta_e, \quad (13e) \]

\[ T_{i1} = T_0 - \frac{\beta}{1 + \beta} (T_0 - T_{\infty}) \text{erfc}\eta_i, \quad (13i) \]

where

\[ \beta = \left( \frac{a_e}{a_i} \right) \left( \frac{\lambda_i}{\lambda_e} \right)^{1/2} \quad (14) \]

and

\[ \eta_e = \frac{y}{2(a_e t)^{1/2}} \]

\[ \eta_i = \frac{|y|}{2(a_i t)^{1/2}}. \]

We notice that (13e) and (13i) are reconfirmed by the Chao’s [9] solutions, adapted for the early times \((\tau << 1)\). As the conditions (5) to (9) are completely satisfied by the first approximations (13e) and (13i), the boundary, initial and interfacial conditions for the equations (12e) and (12i) are:

\( (y > 0) \)

\[ T_{e2}(y, \theta, 0) = 0 \quad (15e) \]

\[ T_{e2}(\infty, \theta, t) = 0 \quad (16e) \]

\[ \frac{\partial T_{e2}}{\partial \theta}(y, 0, t) = \frac{\partial T_{e2}}{\partial \theta}(y, \pi, t) = 0 \quad (17e) \]

\( (y < 0) \)

\[ T_{i2}(y, \theta, 0) = 0 \quad (15i) \]

\[ T_{i2}(-\infty, \theta, t) = 0 \quad (16i) \]

\[ \frac{\partial T_{i2}}{\partial \theta}(y, 0, t) = \frac{\partial T_{i2}}{\partial \theta}(y, \pi, t) = 0 \quad (17i) \]
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with

\[ T_{e2}(0, \theta, t) = T_{i2}(0, \theta, t) \]  
(18)

\[ \lambda_e \frac{\partial T_{e2}}{\partial y}(0, \theta, t) = \lambda_i \frac{\partial T_{i2}}{\partial y}(0, \theta, t). \]  
(19)

Analogously, the equations for the third approximations \((T_{e3}, T_{i3})\) are:

\[ \frac{\partial T_{e3}}{\partial t} - a_e \frac{\partial^2 T_{e3}}{\partial y^2} = 3\bar{z}t \cos \theta \frac{y}{R} \frac{\partial T_{e2}}{\partial y} - \frac{3}{2} \bar{z}t \sin \theta \frac{1}{R} \frac{\partial T_{e2}}{\partial \theta}, \]

\[ \frac{\partial T_{i3}}{\partial t} - a_i \frac{\partial^2 T_{i3}}{\partial y^2} = 3\bar{z}t \cos \theta \frac{y}{R} \frac{\partial T_{i2}}{\partial y} - \frac{3}{2} \bar{z}t \sin \theta \frac{1}{R} \frac{\partial T_{i2}}{\partial \theta}, \]

with the corresponding conditions. Of course, higher-order approximations \(T_{e4}, T_{e5}, ..., T_{i4}, T_{i5}, ...\) can be obtained in a similar manner. However, the complexity of the method of successive approximations increases rapidly as higher approximations are considered.

Now, inserting (13e) and (13i) into (12e) and (12i), then supposing the solutions as follows:

\[ T_{e2} = T_0 - T_\infty \frac{\bar{z}}{1 + \beta} t^2 \cos \theta \quad f_{e2}(\eta_e), \]  
(20e)

\[ T_{i2} = \frac{\beta}{1 + \beta} (T_0 - T_\infty) \frac{\bar{z}}{R} t^2 \cos \theta \quad f_{i2}(\eta_i), \]  
(20i)

we found for the unknown functions \(f_{e2}\) and \(f_{i2}\) the following differential equations:

\[ f''_{e2} + 2\eta_e f'_{e2} - 8f_{e2} = \frac{24}{\sqrt{\pi}} \eta_e \exp(-\eta_e^2), \]  
(21e)

\[ f''_{i2} + 2\eta_i f'_{i2} - 8f_{i2} = \frac{24}{\sqrt{\pi}} \eta_i \exp(-\eta_i^2), \]  
(21i)

with the conditions, issued from (18) and (19):

\[ f_{e2}(0) = \beta f_{i2}(0), \]  
(22e)

\[ f'_{e2}(0) = -f'_{i2}(0). \]  
(22i)
The general solutions of (21e) and (21i) are:

\[ f_{e2}(\eta_e) = K_{e2}g_2(\eta_e) - \frac{2}{\sqrt{\pi}} \eta_e \exp(-\eta_e^2), \quad (23e) \]

\[ f_{i2}(\eta_i) = K_{i2}g_2(\eta_i) - \frac{2}{\sqrt{\pi}} \eta_i \exp(-\eta_i^2), \quad (23i) \]

where \( g_2 \) designates the integral of error function of order 2. Two constants of integration will be calculated by using two conditions (22e) and (22i):

\[ K_{e2} = -\frac{24}{1 + \beta}, \quad (24e) \]

\[ K_{i2} = -\frac{24}{1 + \beta}. \quad (24i) \]

3 On The Growth Of Thermal Boundary Layer

Now, inserting (24e) and (24i) into (23e) and (23i), then adding (20e) and (20i) to (13e) and (13i), the temperature profiles will be determined by (10e) and (10i) in the dimensionless form:

\[ \frac{T_e - T_\infty}{T_0 - T_\infty} = \frac{1}{1 + \beta} \text{erfc}\eta_e - \frac{\bar{z}R^3}{1 + \beta} \frac{\tau^2}{a_e^2} \cos \theta \left[ \frac{24}{1 + \beta} g_2(\eta_e) + \frac{2}{\sqrt{\pi}} \eta_e \exp(-\eta_e^2) \right], \quad (25e) \]

\[ \frac{T_i - T_\infty}{T_0 - T_\infty} = 1 - \frac{\beta}{1 + \beta} \text{erfc}\eta_i - \frac{\bar{z}R^3}{1 + \beta} \frac{\tau^2}{a_i^2} \cos \theta \left[ \frac{24}{1 + \beta} g_2(\eta_i) + \frac{2}{\sqrt{\pi}} \eta_i \exp(-\eta_i^2) \right], \quad (25i) \]

where:

\[ \eta_e = \frac{1}{2\sqrt{\tau R}} \frac{y}{a_e}, \]

\[ \eta_i = \frac{1}{2\sqrt{\tau R}} \frac{|y|}{a_i}. \]
with the dimensionless time:

\[ \tau = \frac{a_e t}{R^2}. \]  

(26)

We note also the physical interpretation of the factor:

\[ \frac{\ddot{z}}{R} t^2 = \frac{\ddot{z}R^3}{a_e^2} \tau^2 = 2 \frac{z}{R}, \]  

(27)

where \( z \) designates the distance covered by the droplet in the time \( \tau \) from the beginning of the motion.

The growth of thermal boundary layers with time for \( \theta = 30^0 \) is shown in Fig. 2 in the case of a benzol droplet \((C_6H_6)\) moving in water at \( 20^0C \) approximately, where:

\[
\begin{align*}
\rho_e &= 1000 \text{kg/m}^3, & \rho_i &= 878, 8 \text{kg/m}^3, \\
\nu_e &= 1, 10^{-6} \text{m}^2/\text{s}, & \nu_i &= 0, 8172.10^{-6} \text{m}^2/\text{s}, \\
a_e &= 0, 1396.10^{-6} \text{m}^2/\text{s}, & a_i &= 0, 0962.10^{-6} \text{m}^2/\text{s}, \\
\lambda_e &= 0, 5876.10^{-3} \text{KW}/\text{0}\text{Cm}, & \lambda_i &= 0, 153.10^{-3} \text{KW}/\text{0}\text{Cm}.
\end{align*}
\]

Using these values, first from (14) we found \( \beta = 3, 2 \), then the temperature profiles are calculated from (25e) and (25i), by arbitrarily assigning \( \frac{\ddot{z}R^3}{a_e^2} = 20 \). Other values may be used but the main features of the finding as described next would not be affected. So, we tested also some other liquid drops (of different petroleum liquids, for instance) moving in water.

### 4 Some Heat Transfer Results

By using (25e), we can determine the local heat flux at the droplet surface:

\[
q = -\lambda_e \frac{\partial T_e}{\partial y}(0, \theta, t)
= \frac{1}{\sqrt{\pi} \sqrt{a_e t}} \frac{\lambda_e}{1 + \beta} \left[ T_0 - T_\infty \right]
+ \frac{1 - \beta}{1 + \beta} \left( \frac{\ddot{z}R^3}{a_e^2} \right) \tau^2 \cos \theta, \]

(28)
as well as the corresponding Nusselt number:

\[
N_u = \frac{2qR}{(T_0 - T_\infty)\lambda_e}
= \frac{2}{\sqrt{\pi}} \frac{1}{1 + \beta \sqrt{a_e}} \left[ 1 + \frac{1 - \beta}{1 + \beta} \left( \frac{\ddot{z}R^3}{a_e^2} \right) \right].
\]  

(29)

Then, the total rate of heat transfer from the droplet to the outside fluid is:

\[
Q = 2\pi R^2 \int_0^\pi q \sin \theta d\theta
= \frac{2\pi R^2}{\sqrt{\pi}} \frac{\lambda_e}{1 + \beta} \left[ I_1 + \frac{1 - \beta}{1 + \beta} \left( \frac{\ddot{z}R^3}{a_e^2} \right) I_2 \right].
\]

(30)

where

\[
I_1 = \int_0^\pi \sin \theta d\theta = 2, \quad I_2 = \int_0^\pi \sin \theta \cos \theta d\theta = 0,
\]
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so:

\[ Q = \frac{4\pi R^2 \lambda_e}{\sqrt{\pi} \sqrt{a_e t}} \left( T_0 - T_\infty \right) 1 + \beta. \]  

(31)

Now, the average Nusselt number is:

\[ \bar{N}_u = \frac{2RQ}{4\pi R^2 (T_0 - T_\infty) \lambda_e} = \frac{2}{\sqrt{\pi} \sqrt{a_e t}} \frac{R}{1 + \beta}, \]  

(32)

which is shown in Fig.3 in the same case as in Fig.2.

Figure 3: Average Nusselt number over droplet surface

Finally, the ratio of the local instantaneous Nusselt number (29) to the average one (32) becomes:

\[ \frac{N_u}{\bar{N}_u} = 1 + \frac{1 - \beta \left( \frac{2R^3}{a_e^2} \right) \tau^2 \cos \theta}{1 + \beta}, \]  

(33)

which, for a given \( \theta \), depends also only on \( \tau \).
5 Concluding Remarks

At small times, say, \( \tau = 0,05 \), the boundary layers remain thin for all cases examined. The growth of the boundary layers at early times is governed by molecular diffusion; convective transport of heat has only a minor contribution. However, as time elapses, the latter assumes a more important role, giving rise to the expected result that, at a given instant, the boundary layer thicknesses increase with increasing \( \theta \).

It is to be noticed that the solutions for the both outer and inner thermal boundary layers, including the third approximations \( (T_{e3}, T_{i3}) \) which we also calculated, confirm all above predictions. But a precise prediction of the thermal heat transfer in accelerated droplet motions would require a more complete understanding of the mechanics of flow separation and the physics and chemistry of the interfacial layers.

References


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O prenosu toplote pri ubrzanom kretanju sferne kapi

UDK 532.526; 533.15

U radu se analiziraju oba tranziciona temperaturska granična sloja, spoljašnji i unutrašnji, u slučaju translatornog jednako-ubrzanog kretanja sferne kapljice iz stanja mirovanja kroz neku drugu viskoznu tećnost, pri velikim brojevima Reynolds-a i Peclet-a. Primenom neviskozne aproksimacije strujnog polja, nadjeno je približno rešenje energijskih jednacina oba strujanja, spoljašnjeg i unutrašnjeg, koje je potom teširano na primeru kapljice benzola u kretanju kroz vodu, sa zadovoljavajućim rezultatima.