Theoret. Appl. Mech., Vol.37, No.1, pp. 49–78, Belgrade 2010

Whole analogy between Daniel Bernoulli solution and direct kinematics solution

Mirjana Filipovic * **Ana Djuric** †

Abstract

In this paper, the relationship between the original Euler-Bernoulli's rod equation and contemporary knowledge is established. The solution which Daniel Bernoulli defined for the simplest conditions is essentially the solution of "direct kinematics". For this reason, special attention is devoted to dynamics and kinematics of elastic mechanisms configuration. The Euler-Bernoulli equation and its solution (used in literature for a long time) should be expanded according to the requirements of the mechanisms motion complexity. The elastic deformation is a dynamic value that depends on the total mechanism movements dynamics. Mathematical model of the actuators comprises also elasticity forces.

Keywords: robot, modeling, elastic deformation, gear, link, coupling, dynamics, kinematics, trajectory planning.

*Mihajlo Pupin Institute, Volgina 15, Belgrade 11000, Serbia, e-mail: mira@robot.imp.bg.ac.yu

†University of Windsor, 401 Sunset Avenue Windsor, Ontario N9B 3P4, Canada,e-mail: adjuric@uwindsor.ca

Nomenclature

EBA	"Euler-Bernoulli approach", assumes the use of Euler-Bernoulli equations which appeared in 1750
LMA	"Lumped-Mass Approach" is a method which defines equation motion of any point of considered mechanism
DOF	degree of freedom
$t(s)$	time
$dt \in R^1(s)$	sample time
$T \in R^1(s)$	whole period time
$p_s = [x\ y\ z\ \psi\ \wp\ \varphi$	Cartesian coordinates
$x_{i,j},\ y_{i,j},\ z_{i,j}$	local coordinate frame, which is set in the base of considered mode
$x_j,\ y_j,\ z_j$	local coordinate frame, which is set in the base of the considered link
$x,\ y,\ z$	basic coordinate frame, which is set in the root of the considered mechanism
$j = 1,\ 2,\ 3,\ \ldots,\ n_i$	serial number of the mode of considered link
$i = 1,\ 2,\ 3,\ \ldots,\ m$	ordinal number of the link
$k = n_1 + n_2 + \ldots + n_m$	whole u number of the modes in considered mechanisms configuration
$M_{i,j} \in R^1(Nm)$	load moment for the mode tip
$\varepsilon_{i,j} \in R^1(Nm)$	bending moment for the mode tip
$\varepsilon_j \in R^{n_i}(Nm)$	bending moments vector for each mode tip considered link
$\varsigma \in R^1(Nm)$	elasticity moment of the gear
$\hat{\#}_{i,j}$	quantities that are related to an arbitrary point of the elastic line of the mode, for example: $\hat{M}_{i,j},\ \hat{x}_{i,j}, \hat{\varepsilon}_{i,j}$
$\#_{i,j}$	quantities that are not designated by "ˆ" are defined for the mode tip, for example: $M_{i,j},\ x_{i,j}, \varepsilon_{i,j}$
$\#_j$	quantities which characterized link
$\#^o$	quantities that are defined desired value
$\vartheta_{i,j} \in R^1(rad)$	bending angle of the considered mode

$\omega_{i,j} \in R^1(rad)$	rotation angle of the tip of the considered mode
$\xi_j \in R^1(rad)$	deflection angle of the gear
$\beta_{i,j} \in R^1(Nm^2)$	flexural rigidity
$\eta_{i,j} \in R^1(s)$	factor which characterizes part of damping in whole flexural characteristics
$H \in R^{kxk}$	inertial matrix
$h \in R^k$	centrifugal, gravitational, Coriolis vector
$F_{uk} \in R^{6x1}(N(Nm))$	dynamic contact force
$m_e \in R^1(kg)$	equivalent mass
$b_e \in R^1(N/(m/s))$	equivalent damping
$k_{a1} \in R^1(N/m)$	environment rigidity
μ	friction coefficient
J_e	Jacobian matrix mapping the effect of the contact force
$T_{sti,j} \in R^1(m)$	stationary part of flexible deformation caused by stationary forces that vary continuously over time
$T_{toi,j} \in R^1(m)$	oscillatory part of flexible deformation
$a_{i,j} \in R^1(m)$	commonly normal distance between j-th and $j+1$-th joints
$\alpha_{i,j} \in R^m(rad)$	angle between the axes z_{j-1} and z_j about axe x_j
$d_{i,j} \in R^1\,(m)$	distance between normal l_{j-1} and l_j along axe of j-th joint
$R_j(\Omega)$	rotor circuit resistance
$i_j(A)$	rotor current
$C_{Ej}(V/(rad/s))$	proportionality constants of the electromotive force
$C_{Mj}(Nm/A)$	proportionality constants of the moment
$B_{uj}(Nm/(rad/s))$	coefficient of viscous friction
$I_j(kgm^2)$	inertia moments of the rotor and reducer
S_j	expression defining the reducer geometry
$\bar{\theta}_j(rad)$	dynamics of motor motion
$l_{i,j} \in R^1(m)$	length of each mode
$r_{i,j} \in R^1(m)$	flexure
$\kappa \in R^1\,(m)$	spatially distance
λ	trajectory mark
$m \in R^1\,(kg)$	mass

$\hat{m}_{el\,i,j}$(kg / m)	equivalent mass of the mode to the considered position on the flexible line
$\hat{J}_{elzz\,i,j}(kgm^2/m)$	equivalent moment of inertia of the mode to the considered position on the flexible line
$E_{km}(Nm)$	kinetic energy
$E_p(Nm)$	potential energy
$\Phi(Nm/s)$	dissipative energy
ϕ	generalized coordinate
$g(m/s^2)$	gravity acceleration
$J_z \in R^1\,(kgm^2)$	inertia moment
$C \in R^{6x6}$	matrix of rigidity
$B \in R^{6x6}$	matrix of damping
$u \in R^1(V)$	control signal
$C_{si,j} \in R^1(kg/s^2)$	characteristics of stiffness of the mode considered link
$B_{si,j} \in R^1(kg/s)$	characteristics of damping of the mode considered link
$C_\xi \in R^1(Nm/rad)$	characteristics of stiffness of the gear
$B_\xi \in R^1\,(Nm/(rad/s))$	characteristics of damping of the gear

1 Introduction

Modeling of elastic mechanisms was a challenge to researchers in the last four decades. In paper [1] the authors extended the integral manifold approach for the control of flexible joint robot manipulators from the known parameter case to the adaptive case. Paper [2] presented the derivation of the equations of motion for application of mechanical manipulators with flexible links. In [3] the equations were derived using Hamilton's principle and they were nonlinear integro-differential equations. Method of variables separation and Galerkin's approach were suggested in the paper [4] for the boundary-value problem with time-dependent boundary condition. The first detailed presentation of the procedure for creating reference trajectory was given in [5].

Spong [6] defined mathematical model of a mechanism with one degree of freedom (DOF) with one elastic gear in 1987. Based on the same principle, the elasticity of gears was introduced in the mathematical model in this paper, as well as in the papers [7]-[12]. However, it is necessary to point out some essential problems in this domain concerning the introduction of link flexibility in the mathematical model.

In our paper, we did not use "assumed modes technique", proposed by Meirovitch in [14]. We disagree with him. "Assumed modes technique" [14] was used by all authors [15]-[25] in the last 40 years to form Euler Bernoulli equation of beam. Unlike our contemporaries, we form Euler Bernoulli equation but we do not use "assumed modes technique" in our paper.

We believe that the "assumed modes technique" can be useful in some other research areas but it is used in a wrong way in robotics, theory of vibrations and theory of elasticity.

LMA ("Lumped-mass approach") is a method, which defines motion equation at any point of considered mechanism.

EBA ("Euler-Bernoulli approach") assumes the use of Euler-Bernoulli equations, which appeared in 1750. EBA [15]-[25] etc, gives the possibility to analyze a flexible line form of each mode in the course of task realization. The EBA approach is still in the focus of researchers' interest. It was analyzed most often in last decades.

The relationship between LMA and EBA was defined in [10] and [11].

In the period from 1750 when the Euler Bernoulli equation was published until today our knowledge, especially in the robotics, the vibration theory and the elasticity theory, have progressed significantly. Therefore, this paper pointed out the necessity of extension of the Euler-Bernoulli equation from many aspects.

In the previous literature [15]-[25] etc, the general solution of the motion of an elastic mechanism was obtained by considering flexural deformations as transversal oscillations that could be determined by the method of particular integrals of D. Bernoulli.

We consider that any elastic deformation can be presented by superimposing D. Bernoulli's particular solutions of the oscillatory character and stationary solution of the forced character. The elastic deformation is a dynamic value, which depends on the total dynamics of the mechanism movements (cf. [7], [8], [10]-[12]).

The reference trajectory is calculated from the overall dynamic model when the mechanism tip tracks a desired trajectory in a reference regime in the absence of disturbances.

Elastic deformation (of flexible links and elastic gears) is a quantity, which is, at least, partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (of both stiffness and damping) are "known" at least partly and at that level, they can be included into the process of defining the reference motion. Thus defined reference trajectory allows the possibility of applying very simple control laws via PD local feedback loops which ensures reliable tracking of the mechanism tip considered in the space of Cartesian coordinates to the level of known elasticity parameters.

In this paper we propose a mathematical model solution that includes in its root the possibility for simultaneous analyzing both present phenomena – the elasticity of gears and the flexibility of links and the idea originated from [26], but based on new principles.

Our future work should be directed on implementation of gears elasticity and links flexibility on any model of rigid mechanism and also on model of reconfigurable rigid robot as given in [29-30] or any other type of mechanism. The mechanism should be modeled to contain elastic elements and to generate vibrations, which are used for conveying particulate and granular materials in [31].

A supplement to source equations of flexible line is given in Section 2. Procedure of defining the dynamic model of the system under the influence of dynamic environment with all elements of coupling is presented completely in Section 3 as well as with dynamic effects of present forces defined. The created (direct and inverse kinematics) kinematic model of system is shown in Section 4. We presented the analogy between the Euler-Bernoulli equation solutions, which were defined, by Daniel Bernoulli and the procedure of the "direct kinematics" and "inverse kinematics" solutions in Section 5. Section 6 gives an analysis of the dynamics movement of a multiple DOF elastic mechanism with elastic gear and flexible link in the presence of the second mode and environment force. Section 7 is devoted to some concluding remarks.

2 Elastic line source equations interpretation

Equation of the elastic line of beam bending has the following form:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} = 0. \tag{1}$$

General motion solution i.e. the form of transversal oscillations of flexible beams can be found by particular integrals method of D. Bernoulli, i.e.:

$$\hat{y}_{to1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t). \tag{2}$$

(see Fig. 1). Superimposing the above particular solutions (2), any transversal oscillation can be presented in the following form:

$$\hat{y}_{to1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t). \tag{3}$$

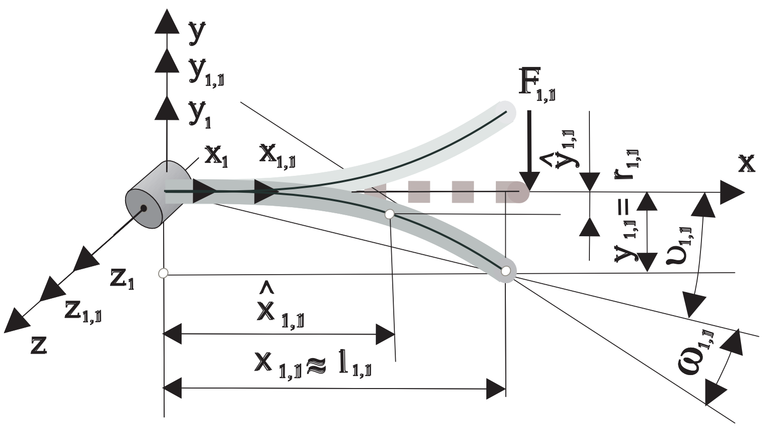

Figure 1: Idealized motion of elastic body according to D. Bernoulli.

Equations (1)-(3) were defined under the assumption that elasticity force was opposed only to the proper inertial force. Besides, it was supposed, according to the definition, that motion in equation (1) was caused by an external force, then suddenly added and finally removed. The solution (2)-(3) of D. Bernoulli supported these assumptions.

Bernoulli presumed that the horizontal position of the observed body was its stationary state (in this case it matched the position x- axis, see Fig. 1). At such presumption, the vibrations happened just around the x- axis. If Bernoulli, at any case, had included the gravity force in its equation (1), the situation would have been more real. Then the stationery body position would not have matched the x- axis position, but the body position would have been little lower and the vibrations would have happened around the new stationery position (as presented in Fig. 2).

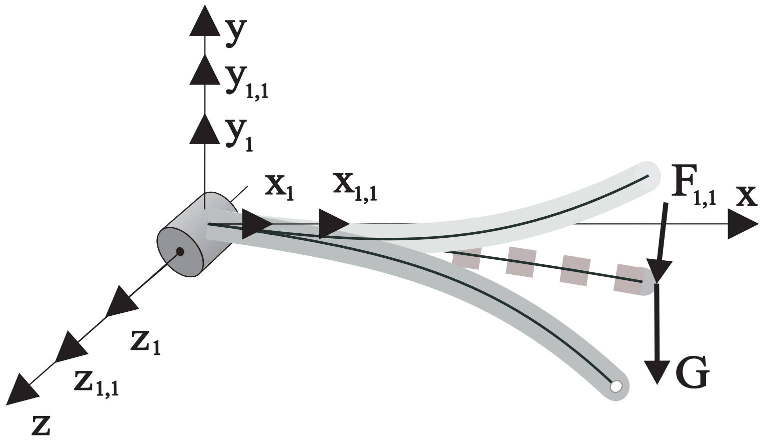

Figure 2: The motion of elastic body in case of gravity force presence.

Equations (1)-(3) need short explanation that, we think, should be assumed, but it could not be found in the literature. Euler and Bernoulli wrote equation (3) based on "vision". They did not define the mathematical model of a link with an infinite number of modes (as presented in Fig. 3), which had a general form of equation (4), but they defined the motion solution (shape of elastic line) of such a link, which was presented in equation (3).

They left the task of link modeling with an infinite number of modes to their successors. Transversal oscillations defined by equation (4) described the elastic beam motion to which we assigned an infinite number of DOFs (modes) and which could be described by a mathematical model composed of an infinite number of equations, in the form:

$$\hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0, \;\; j = 1, 2, ..., j, ...\infty. \tag{4}$$

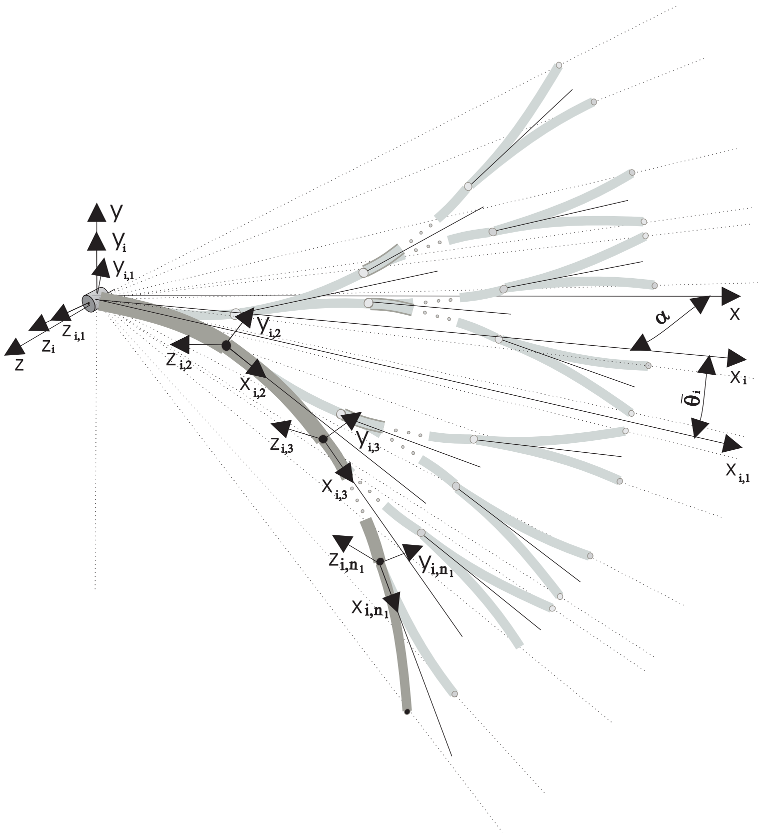

Figure 3: The possible positions of the elastic segment tip.

Dynamics of each mode was described by one equation. The equations in the model (4) were not of equal structure as our contemporaries, authors of numerous works, interpreted them. We think that coupling between involved modes led to the structural diversity among the equations in the model (4). This explanation had the key importance and it was necessary to understand our further discussion.

Under a mode, we considered the presence of coupling between all the modes present in the system. We analyzed the system in which the action of coupling forces (inertial, Coriolis' and elasticity forces) existed between the present modes. This yielded the difference in the structure of Euler-Bernoulli equations for each mode.

The Bernoulli solution (2)-(3) described only partially the nature of real elastic beams motion. To be more precise, it was only one component of motion. Euler-Bernoulli equations (1)-(3) should be expanded from several aspects in order to be applicable in a broader analysis of mechanisms elasticity. Having supplemented these equations with expressions that came out directly from the elastic bodies' motion dynamics, they became more complex.

The motion of the considered mechanism mode was far more complex than the motion of the body presented in Fig. 1. This means that equations describing the mechanism (its elements) must also be more complex than the equations (1)-(3) formulated by Euler and Bernoulli. This fact was overlooked and the original equations were widely used in the literature to describe the mechanism motion. This was very inadequate because valuable pieces of information about the complexity of the elastic mechanism motion were lost that way. Hence, the necessity of expanding the source equations for modeling mechanisms should be emphasized and this should be done in the following way:

- based on the known laws of dynamics, equation (1) was to be supplemented with all forces that participated in the formation of the considered mode bending moment. It was assumed that forces of coupling (inertial, Coriolis, and elastic) between the present modes were also involved which yielded structural difference between equations (1) in the model (4),

- Equations (2)-(3) were to be supplemented with the stationary character of the elastic deformation caused by forces involved.

3 Dynamics

Let us analyze behaviour of the mechanism consisting of elastic gear and flexible link in the presence of the second mode, as depicted in Fig. 4. Tip of mechanism started from position "A" and moves directly to point "B" in predicted time of $T = 2(s)$.

We introduced the presence of the second mode in the analysis of the mechanism behaviour. Angle $\bar{\theta}$is a rotation angle of the motor shaft after the reducer; $\vartheta_{1,1}$ ($\vartheta_{1,2}$) is a bending angle of the first (second) mode of the link; $\omega_{1,1}$ ($\omega_{1,2}$) is a rotation angle of the tip of the first (second) mode (see [32]); ξ is a deflection angle of the gear.

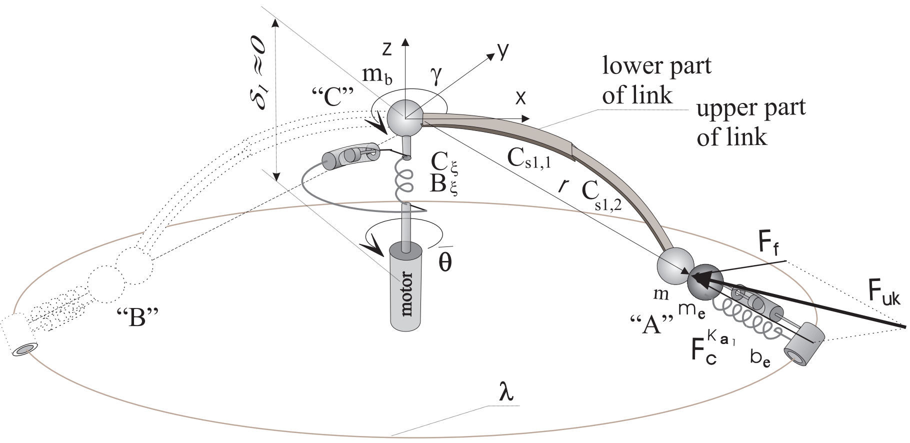

Figure 4: Mechanism.

The relations between the important angles in the horizontal and vertical plane, defined in Fig 5, are given by the following expressions, respectively.

$$q = \gamma + \vartheta_{1,1q}, \quad \gamma = \bar{\theta} + \xi, \quad \delta = \omega_{1,1q} + \vartheta_{1,2\delta}, \quad e = \omega_{1,1m} + \vartheta_{1,2e}. \qquad (5)$$

$$\omega_{1,1q} = \frac{\vartheta_{1,1q}}{2}, \quad \omega_{1,2\delta} = \frac{\vartheta_{1,2\delta}}{2}. \qquad (6)$$

$$\omega_{1,1m} = \frac{\vartheta_{1,1m}}{2}, \qquad \omega_{1,2e} = \frac{\vartheta_{1,2e}}{2} \qquad (7)$$

$q, \delta, \gamma, \quad \vartheta_{1,1m}$ and e are new DH parameters that also encompass the rigidity characteristics ($a_{i,j} = l_{i,j}[m]$, $\alpha_{i,j} = 0^o$ and $d_{i,j} = 0[m]$).

Dynamic model (both the model of flexible line and model of motion of each mode tip) is defined by the equations of motion of elastic mechanism based on classical principles but with the previously introduced new DH parameters, using Lagrange's equations. Generalized coordinates can be defined in an arbitrary way. Literature survey shows that authors always chose flexible deformation of each mode for a generalized coordinate, as well as the motor rotation angle. Obviously, there is no error in such a choice. However, it appears that in such generalized coordinates choice it is not possible to recognize the place where the environment force should be introduced into the final form of the mathematical model. That might be the reason why this final form can also be written in a different way if we carry out regrouping of the model elements. In order to overcome this "problem" we will adopt the following quantities:q,δ,γ,$\vartheta_{1,1m}$, e and $\bar{\theta}$ as generalized coordinates (see Fig. 5).

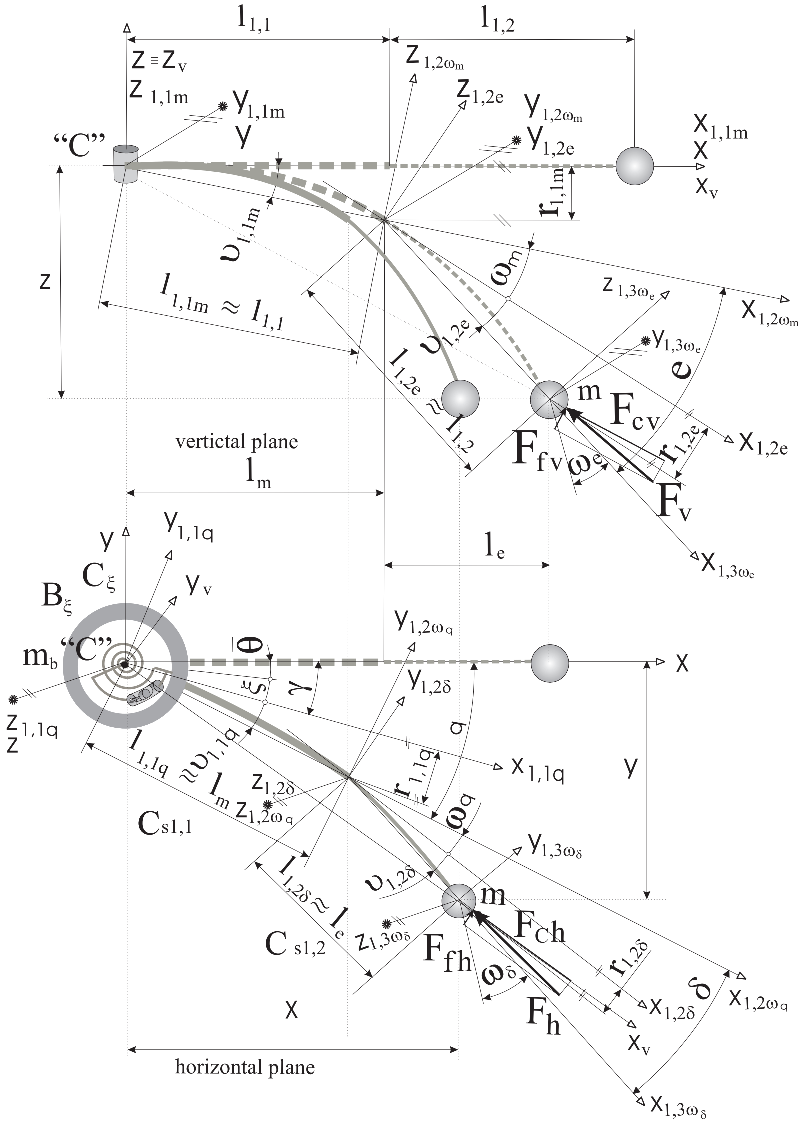

Figure 5: Planar geometry of the mechanism in the horizontal and vertical plane.

We accepted that top of mode was moving continuously on the surface of ball, which radius was $l_{i,j}$ without shortening for each mode.

$$l_{1,1m} \approx l_{1,1}, \;\; l_{1,2e} \approx l_{1,2}, \;\;\; l_{1,1q} \approx l_m, \;\;\; l_{1,2\delta} \approx l_e. \tag{8}$$

If we consider small bending angles and adopt that $tg\,\vartheta_{1,1m} = \dfrac{r_{1,1m}}{l_{1,1}}$, $tg\,\vartheta_{1,2e} = \dfrac{r_{1,2e}}{l_{1,2}}$, $tg\,\vartheta_{1,1q} = \dfrac{r_{1,1q}}{l_m}$, $tg\,\vartheta_{1,2\delta} = \dfrac{r_{1,2\delta}}{l_e}$, then $\vartheta_{1,1m} \approx tg\vartheta_{1,1m}$, $\vartheta_{1,2e} \approx tg\vartheta_{1,2e}$, $\vartheta_{1,1q} \approx tg\vartheta_{1,1q}$, $\vartheta_{1,2\delta} \approx tg\vartheta_{1,2\delta}$, and $l_m = l_{1,1} \cdot \cos\,\vartheta_{1,1m}$, $l_e = l_{1,2} \cdot \cos(\vartheta_{1,1m} + e)$. By applying equations (5)-(7), respectively, we obtain:

$$r_{1,1m} = l_{1,1} \cdot \vartheta_{1,1m}. \tag{9}$$

$$r_{1,2e} = l_{1,2} \cdot (e - \frac{\vartheta_{1,1m}}{2}). \tag{10}$$

$$r_{1,1q} \; = l_{1,1} \cdot \cos\,\vartheta_{1,1m} \cdot (q - \gamma) \tag{11}$$

$$r_{1,2\delta} = l_{1,2} \cdot \cos(\vartheta_{1,1m} + e) \cdot (\delta - \frac{q-\gamma}{2}) \tag{12}$$

Magnitude $r_{1,1m}(r_{1,1q})$ is maximum deflection, i.e. the deflection at the tip of the first mode in vertical (horizontal) plane, while $r_{1,2e}(r_{1,2\delta})$ is maximum deflection, i.e. the deflection at the tip of the second mode in vertical (horizontal) plane.

Component of the whole environment force in the radial direction (see Fig. 4) is: $F_c = (m_e \cdot \ddot{\kappa} + b_e \cdot \dot{\kappa} + F_c^o + k_{a1} \cdot \Delta\kappa)$ whereas the friction force is $F_f = -\mu \dfrac{\dot{k}}{\left|\dot{k}\right|} \cdot F_c$, as in [33]. $F_{uk}^2 = F_c^2 + F_f^2$. But also $F_{uk}^2 = F_h^2 + F_v^2$, where F_v vertical component in $x - z$ plane and F_h horizontal component in x - y plane. $F_h^2 = F_{ch}^2 + F_{fh}^2$, $F_v^2 = F_{cv}^2 + F_{fv}^2$. Follows that $F_h = [F_{ch}\, F_{fh}]^T$, $F_v = [F_{cv}\, F_{fv}]^T$. See Fig. 5.

$\kappa^2 = x^2 + y^2 + z^2$is the distance from the point "C" to the tip of the mechanism, and $\Delta\kappa = (l_{1,1} + l_{1,2}) - \kappa$.

Trajectory, marked with λ on Fig. 4, pertains to the ball surface.

According [32]:

$$\hat{m}_{el\,i,j} = \frac{33}{140} \cdot \bar{w}_{i,j} \cdot \hat{l}_{i,j}, \quad \hat{J}_{elzz\,i,j} = \hat{m}_{el\,i,j} \cdot (\frac{\hat{l}_{i,j}}{2})^2. \tag{13}$$

Kinetic energy of the mechanism presented in Fig. 5 is:

$$\hat{\hat{E}}_{km} = {}^1\!/_2 \cdot \hat{m}_{el\,1,1} \cdot (\hat{l}_{1,1} \cdot \cos\,\hat{\vartheta}_{1,1m})^2 \cdot \dot{q}^2 + {}^1\!/_2 \cdot (m + \hat{m}_{el\,1,2}) \cdot (l_{1,1} \cdot \cos\,\vartheta_{1,1m})^2 \cdot \dot{q}^2 + ... \quad (14)$$

Potential energy of the involved masses is:

$$\hat{\hat{E}}_p = \hat{m}_{el1,1} \cdot \hat{l}_{1,1} \cdot g \cdot \sin\,\hat{\vartheta}_{1,1m} + (m + \hat{m}_{el1,2}) \cdot l_{1,1} \cdot g \cdot \sin\,\vartheta_{1,1m} +$$

$$+ m \cdot l_{1,2} \cdot g \cdot \sin(\,\vartheta_{1,1m} + \,e) + \hat{m}_{el1,2} \cdot \hat{l}_{1,2} \cdot g \cdot \sin(\,\vartheta_{1,1m} + \,\hat{e}). \quad (15)$$

We will express the flexibility moment at any point in the form:
$\hat{\varepsilon}_{i,j} = \beta_{i,j} \cdot \dfrac{\partial^2(\hat{y}_{i,j} + \eta_{i,j} \cdot \dot{\hat{y}}_{i,j})}{\partial\,\hat{x}^2_{i,j}}$.
Now, we should define the potential energy at the tip of each mode:
$E_{pes1,j} = \frac{1}{2} \cdot C_{s1,j} \cdot r^2_{1,j}$.
If we multiply and divide the previous expression by $l^2_{i,j}$, then
$E_{pels1,j} = \dfrac{1}{2} \cdot C_{s1,j} \cdot \dfrac{r^2_{1,j}}{l^2_{1,j}} \cdot l^2_{1,j}$.
By applying equations (9)-(12), respectively, we obtain:

$$E_{pels1,1\,m} = \frac{1}{2} C_{s1,1} (\vartheta_{1,1m})^2 \cdot l^2_{1,1} \quad (16)$$

$$E_{pels1,2\,e} = \frac{1}{2} C_{s1,2} (e - \frac{\vartheta_{1,1m}}{2})^2 \cdot l^2_{1,2}. \quad (17)$$

$$E_{pels1,1\,q} = \frac{1}{2} C_{s1,1} (q - \gamma)^2 \cdot (l^{\cdot}_{1,1} \cos(\vartheta_{1,1m}))^2. \quad (18)$$

$$E_{pels1,2\,\delta} = \frac{1}{2} C_{s1,2} (\delta - \frac{q - \gamma}{2})^2 \cdot (l^{\cdot}_{1,2} \cos(\vartheta_{1,1m} + e))^2. \quad (19)$$

Dissipative energy of the flexible link at the tip of each mode is:

$$\Phi_{els1,1\,m} = \frac{1}{2} B_{s1,1} (\dot{\vartheta}_{1,1m})^2 \cdot l^2_{1,1}. \quad (20)$$

$$\Phi_{els1,2\,e} = \frac{1}{2} B_{s1,2} (\dot{e} - \frac{\dot{\vartheta}_{1,1m}}{2})^2 \cdot l^2_{1,2}. \quad (21)$$

$$\Phi_{els1,1\,q} = \frac{1}{2} B_{s1,1} (\dot{q} - \dot{\gamma})^2 \cdot (l^{\cdot}_{1,1} \cos(\vartheta_{1,1m}))^2. \quad (22)$$

$$\Phi_{els1,2\,\delta} \;= \frac{1}{2} B_{s1,2}(\dot{\delta} - \frac{\dot{q}-\dot{\gamma}}{2})^2 \cdot (l_{1,2}^{\cdot} \cos(\vartheta_{1,1m} + e))^2. \tag{23}$$

Potential energy of the elastic gear is:

$$E_{pel\,\xi} = \frac{1}{2} \cdot C_\xi \cdot \xi^2 = \frac{1}{2} \cdot C_\xi \cdot (\gamma - \bar{\theta})^2. \tag{24}$$

Dissipative energy of the elastic gear is:

$$\Phi_{el\,\xi} = \frac{1}{2} \cdot B_\xi \cdot \dot{\xi}^2 = \frac{1}{2} \cdot B_\xi \cdot (\dot{\gamma} - \dot{\bar{\theta}})^2. \tag{25}$$

$$E_{pel} = E_{pels1,1\,m} + E_{pels1,2\,e} + E_{pels1,1\,q} + E_{pels1,2\,\delta} \; + E_{pel\,\xi}. \tag{26}$$

$$\Phi_{el} = \Phi_{els1,1\,m} + \Phi_{els1,2\,e} + \Phi_{els1,1\,q} + \Phi_{els1,2\,\delta} \; + \Phi_{el\,\xi}. \tag{27}$$

All the angles in the expression for kinetic energy (equation (14)), characterizing flexibility of the links, should also be expressed via generalized coordinates using equations (5)-(7).

Let us define the equation of flexible line of the first mode in horizontal plane. The expressions $\hat{\hat{E}}_{km}$ (14) and $\hat{\hat{E}}_p$ (15) should be defined for the full length of the second mode:

$$l_{1,2} = \int_0^{l_{1,2}} dx_{1,2}, \quad m_{el1,2} = \frac{33}{140} \cdot \bar{w}_{1,2} \cdot l_{1,2}, \quad J_{elzz1,2} = m_{el1,2} \cdot (\frac{l_{1,2}}{2})^2. \tag{28}$$

Thus we obtain the expressions $\hat{E}_{kmel\,1}$ and $\hat{E}_{pel\,1}$. By applying Lagrange's equation respecting the first generalized coordinate q using expressions $\hat{E}_{kmel\,1}$, $\hat{E}_{pel\,1}$, E_{pel} , Φ_{el} we obtain the load moment $\hat{M}_{1,1\,q}$, which represents the sum of all moments that causes the flexible deformation of the first mode in horizontal plane and which is opposed to the flexibility moment $\hat{\varepsilon}_{1,1\,q}$. $\hat{M}_{1,1\,q} + \hat{\varepsilon}_{1,1\,q} = 0$. Magnitude $\hat{M}_{1,1\,q}$ includes also environment force in horizontal plane. This is just procedure for obtaining the Euler-Bernoulli equation of the first mode in the horizontal plane:

$$[\hat{H}_{el1,1} \;\hat{H}_{el1,2} \;\hat{H}_{el1,3} \;0\;0\;0] \cdot \ddot{\phi} + \hat{h}_{el1} \; + \hat{j}^T_{eq} \cdot [F_{ch} \;F_{fh} \;0\;0\;0\;0]^T -$$

$$-\frac{1}{2} \cdot \varepsilon_{1,2\delta} + \beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1q} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1q})}{\partial \,\hat{x}^2_{1,1q}} = 0 \tag{29}$$

We can define the motion of any point on the flexible line of the first mode in horizontal plane by this equation.
$\varepsilon = [\varepsilon_{1,1\,q}\varepsilon_{1,2\,\delta}\varsigma\varepsilon_{1,1\,m}\varepsilon_{1,2\,e}0]^T$
$\varsigma = C_\xi \cdot \xi + B_\xi \cdot \dot{\xi}$ is elasticity force of the gear,
$\varepsilon_{i,j} = (C_{s\,i,j} \cdot \vartheta_{i,j} + B_{s\,i,j} \cdot \dot{\vartheta}_{i,j}) \cdot l_{i,j}^2$ is flexibility moment of the mode,

$$\phi = \begin{bmatrix} q & \delta & \gamma & \vartheta_{1,1m} & e & \bar{\theta} \end{bmatrix}^T.$$

$$\hat{H}_{el1,1} = \hat{m}_{el1,1} \cdot (\hat{l}_{1,1} \cdot \cos\vartheta_{1,1m})^2 + (m + \hat{m}_{el1,2}) \cdot (l_{1,1} \cdot \cos\vartheta_{1,1m})^2 + $$
$$+(m + \hat{m}_{el1,2}) \cdot (l_{1,2} \cdot \cos(\vartheta_{1,1m} + e))^2 +$$
$$+2 \cdot (m + m_{el1,2}) \cdot \hat{l}_{1,1} \cdot \cos\vartheta_m \cdot l_{1,2} \cdot \cos(\vartheta_m + e) \cdot \cos\delta +$$
$$+\frac{9}{4} \cdot \hat{J}_{elzz1,1} + \frac{9}{16} \cdot (J_{zz} + \hat{J}_{elzz1,2})$$

$$\hat{H}_{el1,2} = ..., \quad \hat{H}_{el1,3} = ..., \quad \hat{h}_{el1} = ...$$

In an analogue way we should also define equation of flexible line of the second mode in horizontal plane.

The expressions $\hat{\hat{E}}_{km}(14)$ and $\hat{\hat{E}}_p$ (15) should be defined for the full length of the first mode:

$$l_{1,1} = \int_0^{l_{1,1}} dx_{1,1}, \quad m_{el1,1} = \frac{33}{140} \cdot \bar{w}_{1,1} \cdot l_{1,1}, \quad J_{elzz1,1} = m_{el1,1} \cdot (\frac{l_{1,1}}{2})^2. \quad (30)$$

Thus we obtain expressions $\hat{E}_{kmel\,2}$ and $\hat{E}_{pel\,2}$. By applying Lagrange's equation respecting the second generalized coordinate δ using the expressions $\hat{E}_{kmel\,2}$, $\hat{E}_{pel\,2}$, E_{pel}, Φ_{el} we obtain the load moment $\hat{M}_{1,2\,\delta}$, which represents the sum of all moments that causes the flexible deformation of the second mode in horizontal plane and which is opposed to the flexibility moment $\hat{\varepsilon}_{1,2\,\delta}$. $\hat{M}_{1,2\,\delta} + \hat{\varepsilon}_{1,2\,\delta} = 0$. Magnitude $\hat{M}_{1,2\,\delta}$ includes also environment force in horizontal plane F_h. This is just procedure for obtaining the Euler-Bernoulli equation of the second mode in the horizontal plane:

$$[\hat{H}_{el2,1}\ \hat{H}_{el2,2}\ \hat{H}_{el2,3}\ 0\ 0\ 0] \cdot \ddot{\phi} + \hat{h}_{el2} + \hat{j}_{e\delta}^T \cdot [F_{ch}\ F_{fh}\ 0\ 0\ 0\ 0]^T +$$

$$+\beta_{1,2}\cdot\frac{\partial^2(\hat{y}_{1,2\delta}+\eta_{1,2}\cdot\dot{\hat{y}}_{1,2\delta})}{\partial\,\hat{x}^2_{1,2\delta}}=0 \tag{31}$$

Let us define the equation of flexible line of the first mode in vertical plane. The expressions $\hat{\hat{E}}_{km}$ (14) and $\hat{\hat{E}}_p$ (15) should be defined for the full length of the second mode, equation (28).

Thus we obtain expressions $\hat{E}_{kmel\,1}$ and $\hat{E}_{pel\,1}$. By applying Lagrange's equation respecting the fourth generalized coordinate $\vartheta_{1,1m}$ using the expressions $\hat{E}_{kmel\,1}$, $\hat{E}_{pel\,1}$, E_{pel} , Φ_{el} we obtain the load moment $\hat{M}_{1,1\vartheta_{1,1m}}$, which represents the sum of all moments that causes the flexible deformation of the first mode in vertical plane and which is opposed to the flexibility moment $\hat{\varepsilon}_{1,1\,\vartheta_{1,1m}}.\hat{M}_{1,1\,\vartheta_{1,1m}}+\hat{\varepsilon}_{1,1\,\vartheta_{1,1m}}=0$. Magnitude $\hat{M}_{1,1\,\vartheta_{1,1m}}$ includes also environment force in vertical plane F_v. This is just procedure for obtaining the Euler-Bernoulli equation of the first mode in the vertical plane:

$$[0\ 0\ 0\ \hat{H}_{el4,4}\ \hat{H}_{el4,5}\ 0]\cdot\ \ddot{\phi}+\hat{h}_{el4}+\hat{j}^T_{e\vartheta_{11m}}\cdot[0\,0\,0\,F_{cv}\,F_{fv}\,0]^T-$$

$$-\frac{1}{2}\cdot\varepsilon_{1,2e}+\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1m}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1m})}{\partial\,\hat{x}^2_{1,1m}}=0 \tag{32}$$

In an analogue way we should also define equation of flexible line of the second mode in vertical plane.

The expressions $\hat{\hat{E}}_{km}$ (14) and $\hat{\hat{E}}_p$ (15) should be defined for the full length of the first mode, equation (30).

Thus we obtain expressions $\hat{E}_{kmel\,2}$ and $\hat{E}_{pel\,2}$. By applying Lagrange's equation respecting the fifth generalized coordinate e using the expressions $\hat{E}_{kmel\,2}$, $\hat{E}_{pel\,2}$, E_{pel} , Φ_{el} we obtain the load moment $\hat{M}_{1,2\,e}$, which represents the sum of all moments that causes the flexible deformation of the second mode in vertical plane and which is opposed to the flexibility moment $\hat{\varepsilon}_{1,2\,e}$. $\hat{M}_{1,2\,e}+\hat{\varepsilon}_{1,2\,e}=0$. Magnitude $\hat{M}_{1,2\,e}$ includes also environment force in vertical plane F_v. This is just procedure for obtaining the Euler-Bernoulli equation of the second mode in the vertical plane:

$$[0\ 0\ 0\ \hat{H}_{el5,4}\ \hat{H}_{el5,5}0\ ]\cdot\ \ddot{\phi}+\hat{h}_{el5}+\hat{j}^T_{ee}\cdot[0\,0\,0\,F_{cv}\,F_{fv}\,0]^T\ +$$

$$+\beta_{1,2}\cdot\frac{\partial^2(\hat{y}_{1,2e}+\eta_{1,2}\cdot\dot{\hat{y}}_{1,2e})}{\partial\,\hat{x}^2_{1,2e}}=0 \tag{33}$$

Users are especially interested in the motion of the mode tip. Inertial forces (own and the coupled ones of other modes), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to the relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the environment force act at this point, whereat the effect of the latter on the motion of the considered link is transferred through the Jacobian matrix.

The equation of motion of the forces involved at any point of the elastic line of first mode in horizontal plane, including the point of the first mode tip, can be defined in the following way.

The expressions $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_{p}$ should be defined for the full length of the first mode $l_{1,1} = x_{1,1}$and for the full length of the second mode $l_{1,2} = x_{1,1}$. The expressions E_{kmel} and E_{pel} are derived this way. The equation of the motion of the tip point of considered elastic line of the first mode in horizontal plane is obtained by applying Lagrange's equation with respect to the generalized coordinate q and using the expressions E_{kmel}, E_{pel}, E_{pel} , Φ_{el} .

$$\begin{gathered}[H_{el1,1}\ H_{el1,1}\ H_{el1,1}\ 0\ 0\ 0] \cdot\ \ddot{\phi} + h_{el1} + \\ + j_{eq}^{T} \cdot [F_{ch}\, F_{fh}\, 0\, 0\, 0\, 0]^{T} - \frac{1}{2} \cdot \varepsilon_{1,2\delta} + \varepsilon_{1,1q} = 0 \end{gathered} \tag{34}$$

Following the same procedure by applying Lagrange's equation to expressions E_{kmel}, E_{pel}, E_{pel} , Φ_{el} with respect to other generalized coordinates δ,γ,$\vartheta_{1,1m}$, e , the equations of motion at the tip point of the considered elastic line considered mode respectively are obtained.

$$[H_{el2,1}\ H_{el2,2}\ H_{el2,3}\ 0\ 0\ 0] \cdot\ \ddot{\phi} + h_{el2} + j_{e\delta}^{T} \cdot [F_{ch}\, F_{fh}\, 0\, 0\, 0\, 0]^{T} + \varepsilon_{1,2\delta} = 0 \tag{35}$$

$$[H_{el3,1}\ H_{el3,2}\ H_{el3,3}\ 0\ 0\ 0] \cdot\ \ddot{\phi} + h_{el3} + \frac{1}{2}\varepsilon_{1,2\delta} - \varepsilon_{1,1q} + \varsigma = 0 \tag{36}$$

$$[0\ 0\ 0\ H_{el4,4}\ H_{el4,5}\ 0] \cdot\ \ddot{\phi} + h_{el4} + j_{e\vartheta_{11m}}^{T} \cdot [0\, 0\, 0\, F_{cv}\, F_{fv}\, 0]^{T} - \frac{1}{2} \cdot \varepsilon_{1,2e} + \varepsilon_{1,1m} = 0 \tag{37}$$

$$[0\ 0\ 0\ H_{el5,4}\ H_{el5,5} 0\ ] \cdot\ \ddot{\phi} + h_{el5} + j_{ee}^{T} \cdot [0\, 0\, 0\, F_{cv}\, F_{fv}\, 0]^{T} + \varepsilon_{1,2e} = 0 \tag{38}$$

By applying Lagrange's equation with respect to the sixth generalized coordinate $\bar{\theta}$, we obtain the equation of the motor motion:

$$u = R \cdot i + C_E \cdot \dot{\bar{\theta}}, \qquad C_M \cdot i = I \cdot \ddot{\bar{\theta}} + B \cdot \dot{\bar{\theta}} - S \cdot \varsigma. \tag{39}$$

Equations (34)-(39) that we should write in the matrix form obtain the mathematical model of the system depending on the selected generalized coordinates q, δ, γ,$\vartheta_{1,1m}$, e, $\bar{\theta}$:

$$U = H \cdot \ddot{\phi} + h + C \cdot \phi + B \cdot \dot{\phi} + J_e^T \cdot F_p^T. \tag{40}$$

$F_p = [F_{ch}\, F_{fh}\, 0\, F_{cv}\, F_{fv}\, 0]$.

Via equation (40) we can define motions q, δ, γ, $\vartheta_{1,1m}$, e and $\bar{\theta}$ and through them the angle of gear deflection, as well as the bending angle for the tip of each mode in horizontal and vertical plane, but we cannot define the motions of particular points on the flexible line of the present modes.

Remark: Equations (29), (31), (32), (33) can not be equated to the equations (34), (35), (37), (38), respectively because they are equations of different type. Equations (29), (31), (32), (33) are Euler-Bernoulli equations, while the equations (34), (35), (37), (38) are equations of motion at the point of the tip of the considered mode (LMA). The equations of the model (40) are also equations of motion at a certain point. The system (40) consists of the equations of the same type. Through them we can analyze the motion of the mechanism tip. $H \in R^{6x6}$.

$$\begin{gathered} H_{1,1} = m_{el1,1} \cdot (l_{1,1} \cdot \cos\vartheta_m)^2 + (m + m_{el1,2}) \cdot (l_{1,1} \cdot \cos\vartheta_m)^2 + \\ (m + m_{el1,2}) \cdot (l_{1,2} \cdot \cos(\vartheta_m + e))^2 + \\ 2 \cdot (m + m_{el1,2}) \cdot l_{1,1} \cdot \cos\vartheta_m \cdot l_{1,2} \cdot \cos(\vartheta_m + e) \cdot \cos\delta + \\ \frac{9}{4} \cdot J_{elzz1,1} + \frac{9}{16} \cdot (J_{zz} + J_{elzz1,2}) \\ H_{1,2} = ..., \quad H_{1,3} = ..., \end{gathered}$$

$h \in R^{1x6}$, $C \in R^{6x6}-$ is the matrix of rigidity,
$B \in R^{6x6}-$ is the matrix of damping.
Control is denoted by: $U = [0\ 0\ 0\ 0\ 0\ u]^T$.

$$u = K_{lp} \cdot (\bar{\theta}^o - \bar{\theta}) + K_{lv} \cdot (\dot{\bar{\theta}}^o - \dot{\bar{\theta}}). \tag{41}$$

$$J_e = \begin{bmatrix} J_{e1,1h} & J_{e1,2h} & 0 & 0 & 0 & 0 \\ J_{e2,1h} & J_{e2,2h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{e1,1v} & J_{e1,2v} & 0 \\ 0 & 0 & 0 & J_{e2,1v} & J_{e2,2v} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J_{eq} \\ J_{e\delta} \\ 0_6 \\ J_{e\vartheta_{11m}} \\ J_{ee} \\ 0_6 \end{bmatrix}$$

- is the Jacobian matrix.

4 Kinematics

A geometric link between these characteristics (internal coordinates q, δ, γ, $\vartheta_{1,1m}$, e and $\bar{\theta}$) and the space of Cartesian coordinates (external coordinates) $p_s = \begin{bmatrix} x & y & z & \psi & \wp & \varphi \end{bmatrix}^T$ was defined as so-called "direct kinematics". In this case (see Fig. 5):

$$\begin{gathered} x = l_m \cdot \cos q + l_e \cdot \cos(q+\delta) \\ y = l_m \cdot \sin q + l_e \cdot \sin(q+\delta) \\ x_v = l_{1,1} \cdot \cos \vartheta_{1,1m} + l_{1,2} \cdot \cos(\vartheta_{1,1m} + e) \\ z_v = l_{1,1} \cdot \sin \vartheta_{1,1m} + l_{1,2} \cdot \sin(\vartheta_{1,1m} + e) \\ z \equiv z_v \end{gathered} \tag{42}$$

From equation (42) $\dot{x}$, $\dot{y}$, $\dot{z}$ and $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ can be calculated.

The Jacobi matrix for a manipulator with elastic joints and links maps the velocity vector of the external coordinates $\dot{p}_s$ into the velocity vector of internal coordinates $\dot{\Phi}$:

$$\dot{\Phi} = J_e^{-1}(\Phi) \cdot \dot{k}_s. \tag{43}$$

Where $\dot{k} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & \dot{\psi} & \dot{\wp} & \dot{\varphi} \end{bmatrix}^T$ defines the velocity of a given point of the mechanism in the Cartesian coordinates, whereas $\dot{\phi} = \begin{bmatrix} \dot{q} & \dot{\delta} & \dot{\gamma} & \dot{\vartheta}_{1,1m} & \dot{e} & \dot{\bar{\theta}} \end{bmatrix}^T$ defines the velocity vector of internal coordinates.

We form elements of Jacobi matrix J_e in our example only for each plane. In $x-y$ plane we have J_{eh} Jacobi matrix. See Fig. 5.

$$J_{eh} = \begin{bmatrix} J_{e1,1h} \cdot & J_{e1,2h} \\ J_{e2,1h} & J_{e2,2h} \end{bmatrix} = \begin{bmatrix} -(l_m \cdot \sin(q+\delta) + l_e \cdot \sin q) & -l_m \cdot \sin(q+\delta) \\ l_m \cdot \cos(q+\delta) + l_e \cdot \cos q & l_m \cdot \cos(q+\delta) \end{bmatrix}. \tag{44}$$

When mechanism is at rest, elastic deformation is raised only by the gravitation force.

5 The connection between the Euler-Bernoulli equation solution and so-called "direct kinematics solution"

The authors Euler and Bernoulli defined the equation (1) under simple and almost idealized conditions and its solution (2) was only the consequence of these ultimately simplified conditions.

The Euler-Bernoulli equation (4) in vector form was defined under ultimately simplified conditions (which do not diminish its significance) and its solution was defined by Daniel Bernoulli and presented by the equation (3) in the original form.

The solution of Euler-Bernoulli equation, defined by Daniel Bernoulli in the form of equation (2), i.e. (3), was, actually, the definition of the position of any point on the segment elastic line (including the tip points) in any selected moment, which was completely analogue to the "direct kinematics" solution. We say *"kinematics"* in the terms of the rigid mechanisms because in that case that really is kinematics. However, when the segment elasticity is present, then the elastic deformation values, which are by their nature dynamic values, take part in the definition of the position and orientation of every point on the mechanism elastic line. In addition, for that reason, in order to keep the familiar terminology, in future we will imply that solving of the "direct *kinematics*" in the elastic mechanisms means the presence of the elastic deformations.

In order to come to this important conclusion we had previously to do the following:

- to extend significantly the original Euler-Bernoulli equation (1) in both form and content by adding all the forces that took part in creation of elastic line of every elastic element mode and to bring them in the form given with the equations (29), (31), (32), (33) in Example,
- to define the connection between EBA- these equations are (29), (31), (32), (33) and LMA- these equations are (34), (35), (37), (38),
- to define properly new form of the motor mathematic model that had the form of equation (39),

- to define its movement solution that had the form of the equation (42), on the basis of the total mechanism elastic model, which was defined in the classic form by equations (34), (35), (36), (37), (38) and equation (39).

That way we presented the analogy between:

the Euler-Bernoulli equation solutions which were defined by Daniel Bernoulli by the equation (3) in the original form	and	the procedure of the "direct kinematics" solutions by equation (42).

The analogy between the Euler-Bernoulli equation and its solution and modern knowledge was presented that way.

6 The simulations example

The reference trajectory is defined in purely kinematics way i.e. geometric and now in the presence of the elasticity elements we can include the elastic deformation values at the reference level i.e. at the level of knowing the elasticity characteristics during the reference trajectory defining.

A mechanism starts from the point "A" (Fig. 4) and moves towards the point "B" in the predicted time $T = 2\,[s]$. The adopted velocity profile is trapezoidal, with the period of acceleration/deceleration of $0.2 \cdot T \cdot dt = 0.000053335[s]$. Elastic deformation is a quantity, which is, at least partly, encompassed by the reference trajectory. The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link.

$C_\xi = 0.2 \cdot C_\xi^o, \qquad B_\xi = 0.2 \cdot B_\xi^o,$
$C_{s1,1} = 0.99 \cdot C_{s1,1}^o, \qquad B_{s1,1} = 0.99 \cdot B_{s1,1}^o,$
$C_{s1,2} = 0.99 \cdot C_{s1,2}^o, \qquad B_{s1,2} = 0.99 \cdot B_{s1,2}^o.$

The only disturbance in the system is the partial lack of the knowledge of all flexibility characteristics.

As it can be seen from Fig. 6 in its motion from point "A" to point "B", the mechanism tip tracks the reference trajectory in the space of Cartesian coordinates.

Since a position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of position from the reference level (see Fig. 7).

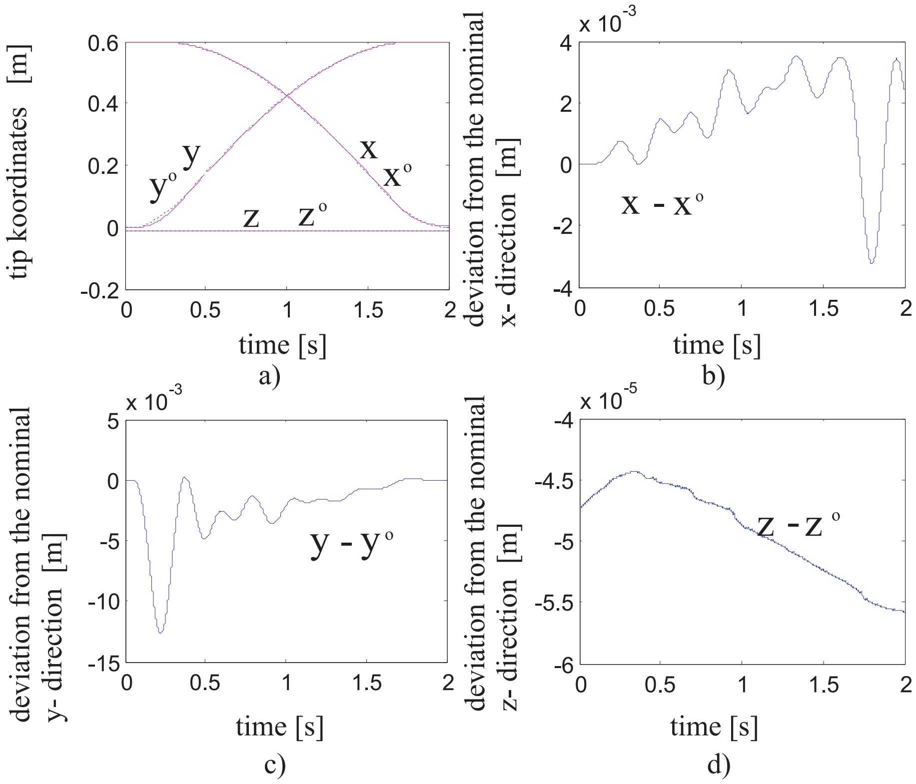

Figure 6: The tip coordinates and the position deviation from the reference level.

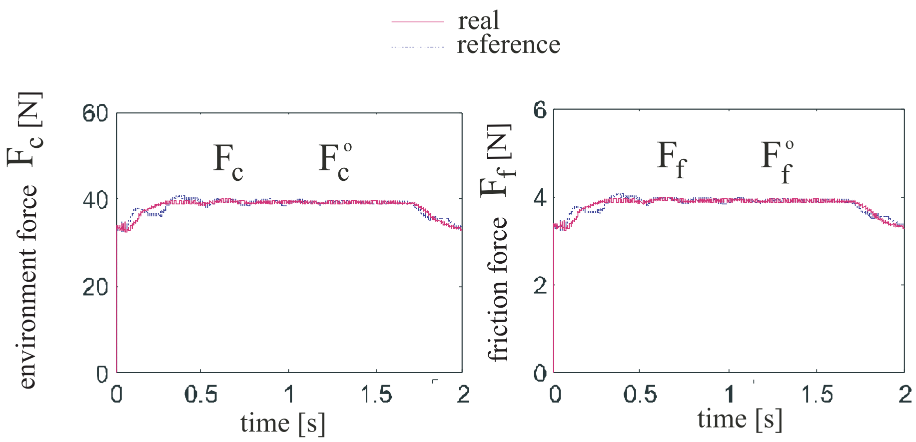

Figure 7: The environment force dynamics.

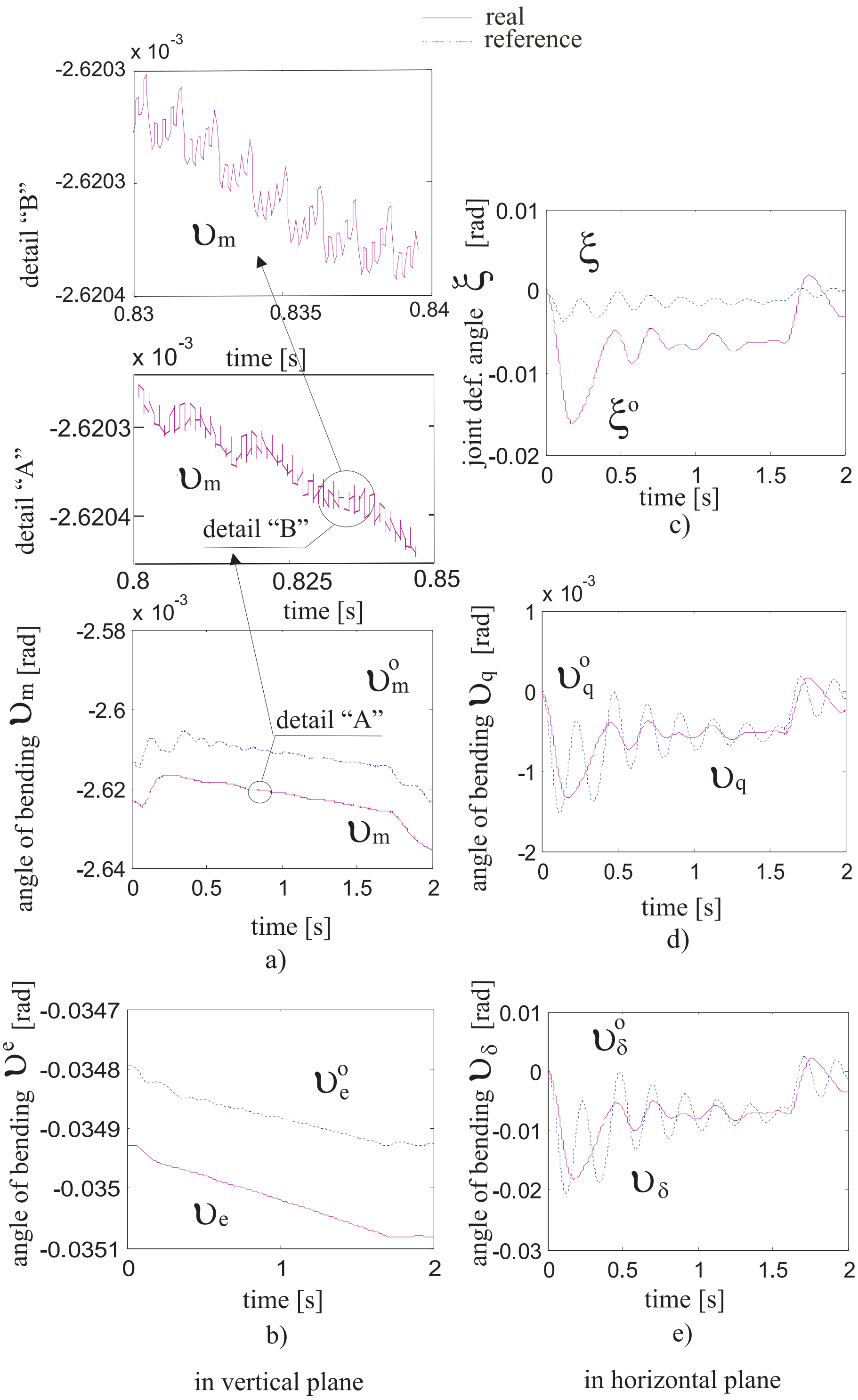

Figure 8: The elastic deformations.

The elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (the first mode) $\vartheta_{1,1m}$ and the angle of bending of the upper part of the link (the second mode) $\vartheta_{1,2e}$, as well as elastic deformations taking place in the horizontal plane, the angle of bending of the lower part of the link (the first mode) $\vartheta_{1,1q}$, the angle of bending of the upper part of the link (the second mode) $\vartheta_{1,2\delta}$ and the deflection angle of gear ξ were given in Fig. 8.

The rigidity of the second mode is about ten times lower compared to that of the first mode. Then it is logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

A more significant lack of knowledge of gear flexibility characteristics causes larger deviations of this quantity from the reference in the course of mechanism task realization.

Let us present the special significance of results from Fig. 8a. This figure exhibits the wealth of different amplitudes and circular frequencies of the present modes of elastic elements. We have vibrations within vibrations. This confirms that we modeled all elastic elements as well as high harmonics (in this case two harmonics of considered link).

7 Conclusions

It should be pointed out that the elastic deformation is the consequence of the total mechanism dynamics, which is essentially different from widely used method that implies the adaptation of the "assumed modes technique".

The analogy was defined between the solution of the Euler-Bernoulli equation, which Daniel Bernoulli defined in the original form and "the direct kinematics solution".

With fundamental approach to analysis of flexibility of the complex mechanism, a wide field of working on analyzing and modeling of complex mechanical construction as well as implementation of different control of laws was opened. All this was presented for a relatively "simple" mechanism that offered the possibility of analyzing the phenomena involved.

The formed mathematical model of robot mechanism with elastic segment in the presence of higher harmonics (of the second mode) served as a basis for the formation of the Software package TMODES. All presented simulations were the result of the developed Software package TMODES.

Through the analysis and modeling of an elastic mechanism we attempted to give a contribution to the development of this area.

References

[1] F.Ghorbel and W.M.Spong, Adaptive Integral Manifold Control of Flexible Joint Robot Manipulators, *IEEE International Conference on Robotics and Automation*, Nice, France, (May 1992).

[2] K.H.Low, A Systematic Formulation of Dynamic Equations for Robot Manipulators with Elastic Links, *J Robotic Systems*, Vol. 4, No. 3, (June 1987), 435-456.

[3] K.H.Low and M.Vidyasagar, A Lagrangian Formulation of the Dynamic Model for Flexible Manipulator Systems, *ASME J Dynamic Systems, Measurement, and Control*, Vol. 110, No. 2, (Jun 1988), pp.175-181.

[4] K.H.Low, Solution Schemes for the System Equations of Flexible Robots, *J. Robotics Systems*, Vol. 6, No. 4, (Aug 1989), 383-405.

[5] E.Bayo, A Finite-Element Approach to Control the End-Point Motion of a Single-Link Flexible Robot, J. of Robotic Systems Vol. 4, No 1, (1987), 63-75.

[6] W.M.Spong, Modelling and control of elastic joint robots, *ASME J.of Dynamic Systems, Measurement and Control,* **109**, (1987), pp.310-319.

[7] M.Filipovic and M.Vukobratovic, Modeling of Flexible Robotic Systems, *Computer as a Tool, EUROCON 2005, The International Conference,* Belgrade, Serbia and Montenegro, Volume 2, (21-24 Nov. 2005), 1196 - 1199.

[8] M.Filipovic and M.Vukobratovic, Contribution to modeling of elastic robotic systems", *Engineering & Automation Problems, International Journal,* Vol. 5, No 1, (September 23. 2006), 22-35.

[9] M.Filipovic, V.Potkonjak and M.Vukobratovic, Humanoid robotic system with and without elasticity elements walking on an immobile/mobile platform, *Journal of Intelligent & Robotic Systems, International Journal,* Volume 48, (2007), 157 - 186.

[10] M.Filipovic and M.Vukobratovic, Complement of Source Equation of Elastic Line, *Journal of Intelligent & Robotic Systems, International Journal,* Volume 52, No 2, (June 2008), 233 - 261.

[11] M.Filipovic and M.Vukobratovic, Expansion of source equation of elastic line, *Robotica, International Journal*, (November 2008), 1 - 13.

[12] M.Filipovic, New form of the Euler-Bernoulli rod equation applied to robotic systems, *Theoretical and Applied Mechanics, International Journal,* Vol. 35, No. 4, (2008) 381-406.

[13] M.Filipovic, Euler-Bernoulli Equation Today, IROS 2009, IEEE/RSJ International Conference on Intelligent Robots and Systems, St. Louis, MO, USA (October 2009).

[14] L.Meirovitch, *Analytical Methods in Vibrations...* New York, Macmillan, (1967).

[15] A.De Luca, Feedforward/Feedback Laws for the Control of Flexible Robots, *IEEE Int. Conf. on Robotics and Automation*, (April 2000), 233-240.

[16] F.Matsuno, K.Wakashiro and M.Ikeda, Force Control of a Flexible Arm, *International Conference on Robotics and Automation*, (1994), 2107-2112.

[17] F.Matsuno and T.Kanzawa, Robust Control of Coupled Bending and Torsional Vibrations and Contact Force of a Constrained Flexible Arm, *International Conference on Robotics and Automation*, (1996), 2444-2449.

[18] D.Surdilovic and M.Vukobratovic, One method for efficient dynamic modeling of flexible manipulators, *Mechanism and Machine Theory*, Vol. 31, No. 3, (1996), 297-315.

[19] J. Cheong, W.K.Chung and Y.Youm, PID Composite Controller and Its Tuning for Flexible Link Robots, *Proceedings of the 2002 IEEE/RSJ, Int. Conference on Intelligent Robots and Systems EPFL*, Lausanne, Switzerland, (October 2002).

[20] F.Matsuno and Y.Sakawa, Modeling and Quasi-Static Hybrid Position/Force Control of Constrained Planar Two-Link Flexible Manipulators, *IEEE Transactions on Robotics and Automation*, Vol. 10, No 3, (June 1994).

[21] J-S.Kim, K.Siuzuki and A.Konno, Force Control of Constrained Flexible Manipulators, *International Conference on Robotics and Automation*, (April 1996), 635-640.

[22] A.De Luka and B.Siciliano, Closed-Form Dynamic Model of Planar Multilink Lightweight Robots, IEEE Transactions on *Systems, Man, and Cybernetics*, Vol. 21, (July/August 1991), 826-839.

[23] H.Jang, H.Krishnan and M.H.Ang Jr., A simple rest-to-rest control command for a flexible link robot, *IEEE Int. Conf. on Robotics and Automation*, (1997), 3312-3317.

[24] J.Cheong, W.Chung and Y.Youm, Bandwidth Modulation of Rigid Subsystem for the Class of Flexible Robots, *Proceedings Conference on Robotics &Automation*, San Francisco, (April 2000), 1478-1483.

[25] S.E.Khadem and A.A.Pirmohammadi, Analytical Development of Dynamic Equations of Motion for a Three-Dimensional Flexible Link Manipulator With Revolute and Prismatic Joints, IEEE Transactions on Systems, Man and Cybernetics, part B, Cybernetics, Vol. 33, No. 2, (April 2003).

[26] W.J.Book and M.Majette, Controller Design for Flexible, Distributed Parameter Mechanical Arms via Combined State Space and Frequency Domain Techniques, Trans ASME J. Dyn. Syst. Meas. And Control, 105, (1983), 245-254.

[27] W.J. Book, Recursive Lagrangian Dynamics of Flexible Manipulator Arms, International Journal of Robotics Research. Vol. 3. No 3, (1984).

[28] W.J. Book, Analysis of Massless Elastic Chains with Servo Controlled Joints. Trans ASME J. Dyn.Syst. Meas. And Control 101, 187-192 (1979).

[29] A.M. Djuric, W.H. ElMaraghy, E.M. ElBeheiry, Unified integrated modelling of robotic systems, NRC International Workshop on Advanced Manufacturing, London, Canada (June 2004).

[30] A.M. Djuric, W.H. ElMaraghy, Unified Reconfigurable Robots Jacobian. Proc. of the 2nd Int. Conf. on Changeable, Agile, Reconfigurable and Virtual Production. 811-823 (2007).

[31] Z.Despotovic, Z.Stojiljkovic, Power Converter Control Circuits for Two-Mass Vibratory Conveying System with Electromagnetic Drive: Simulations and Experimental Results IEEE Translation on Industrial Electronics. 54, I, 453-466 (February 2007).

[32] J. W. Strutt (Lord Rayleigh), The Theory of Sound, Vol.. II, paragraph 186, Mc. Millan & Co. London and New York (1894-1896).

[33] V.Potkonjak and M.Vukobratovic, Dynamics in Contact Tasks in Robotics, Part I General Model of Robot Interacting with Dynamic Environment, *Mechanism and Machine Theory*, Vol. 33, (1999).

Submitted on November 2009, revised on February 2010.

Puna analogija izmedju rešenja Daniela Bernoulli-a i rešenja direktne kinematike

U ovom radu je uspostavljena veza izmedju originalne Euler-Bernoulli's jednačine i savremenih znanja. Rešenje koje je definisao Daniel Bernoulli za pojednostavljene uslove je u suštini rešenje ,,direktne kinematike". Iz tih razloga posebna pažnja je posvećena dinamici i kinematici konfiguracija elastičnih mehanizama. Euler-Bernoulli jednačina a takodje i njeno rešenje (korišćeno u literaturi dugi niz godina) treba proširiti prema zahtevima složenosti kretanja mehanizma. Elastična deformacija je dinamička veličina koja zavisi od ukupne dinamike kretanja mehanizma. Matematički model aktuatora sadrži takodje sile elastičnosti.

doi:10.2298/TAM1001049F Math.Subj.Class.: 68T40