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A new approach to Bäcklund transformations for longitudinal dispersion of miscible fluid flow through porous media in oil reservoir during secondary recovery process

Ramakanta Meher* **M.N.Mehta†**
S.K.Meher‡

Abstract

In this paper a theoretical model has developed for the dispersion problem in porous media in which the flow is one dimensional and the average flow is unsteady. The Solution of the dispersion problem is presented by means of a new approach to Bäcklund transformations of nonlinear partial differential equations.

Keywords: Miscible fluid, Burger equation, Bäcklund transformation, dispersion, concentration, flows in porous media.

*Department of Mathematics, S.V.National Institute of Technology, Surat-395007, India, e-mail: meher_ramakanta@yahoo.com
†Department of Mathematics, S.V.National Institute of Technology, Surat-395007, India, e-mail: mnm@ashd.svnit.ac.in
‡Department of Mathematics, S.V.National Institute of Technology, Surat-395007, India

Nomenclature

C_0	Initial concentration of solute in liquid phase
C	concentration of solute in liquid phase
ρ	density of the fluid
$\bar{v}$	pore seepage velocity
D	dispersion coefficient based on u
t	time(s)
x	linear coordinate (m)

1 Introduction

The soil plays one of the most important roles in the hydrological cycle. It is a three-phase porous medium, where the phases are particles of ground rock or clay, water and a water vapour air combination. The region of the soil that is unsaturated is known as the vadose zone (or simply the unsaturated zone), and it is in this region where the most interesting nonlinear hysteretic behavior is observed.

Soil moisture is more generally considered within the context of hydrology, where it represents the immediate store of infiltrating rainfall, before it either evapotranpires or contributes to groundwater recharge. This term is used in hydrogeology, soil sciences and soil mechanics. In saturated ground water aquifers, all available pore spaces are filled with water. Above a capillary fringe pore spaces have air in them too.

A common example is the mixing of viscous fluids in chemical engineering applications. Similar processes encountered in environmental problems such as the spreading of aqueous or non aqueous pollutants following an accidental discharge. the mixing of these pollutants with surrounding water flows generally includes underground porous layers.

Let us analyze the generic case of two fluids in contact flowing through a porous medium: Mixing is almost always associated to the random walk of fluid (or tracer) particles through the disordered structure of the pore volume and thermal molecular agitation is dominant only at very low mean flow velocities. The steps of the random walk are much larger than those of thermal Brownian motion so that the corresponding spreading scale and the width of the dispersion front is correspondingly increased. Of course, the minimum size of heterogeneities of the mixture obtained in this way is also larger; however, if the medium is adequately homogeneous, this size is of the order of the grain diameter so that molecular diffusion can generally complete the mixing. One generally characterizes the relative influence of the various spreading mechanisms by the Peclet number

$$P_e = \frac{U_d}{D_m}$$

where d is the characteristic size of the grains of the medium, U is the mean flow velocity, and D_m is the molecular diffusion coefficient for the species considered; Pe characterizes the relative influence of convective spreading effect due to the disorder of the medium and of molecular diffusion. For a homogeneous porous packing in which the correlation length of the velocity field is of the order of the grain size, the value $Pe = 1$ corresponds to the boundary between dominant convective ($Pe > 1$) and diffusive ($Pe < 1$) spreading mechanisms.

As far back as 1930, it was shown experimentally that there is a hysteretic effect in the relationship of the moisture content and the capillary pressure in soils that are not fully saturated. This effect can be quite strong in certain types of soils, and therefore it is desirable to incorporate hysteresis in the dynamical models describing flows of water through the soil. It is believed that the hysteresis of soil-water is rate-independent when considered on the time-scales of water flow.

The problem of miscible displacement can be observed in coastal areas, where the fresh water beds are gradually displaced by sea water. These day's efforts are being made by the environmentalist to dispose the atomic waste products born from nuclear reactor and dumped inside the ground by using the same phenomenon of displacement.

Among Many flow problems in porous media, one involves fluid mixtures called miscible fluids. A miscible fluid is a single phase fluid consisting of several completely dissolved homogenous fluid species, a distinct fluid-fluid interface doesn't exist in a miscible fluid. The flow of miscible fluid is an important topic in petroleum industry; an enhanced recovery technique in oil reservoir involves injecting a fluid (solvent) that will dissolve the reservoir's oil.

In a miscible displacement process a fluid is displaced in a porous medium by another fluid that is miscible with the first fluid. Miscible displacement in porous media plays a prominent role in many engineering and science fields such as oil recovery in petroleum engineering, contamination of ground water by waste product disposed under ground movement of mineral in the soil and recovery of spent liquors in pulping process.

The key benefit of this research is to improved conceptual models how all contaminants migrate through heterogeneous, variably-saturated, porous media. Research activities are driven by the hypothesis that the reactivity of variably saturated porous media is dependent on the moisture content of the medium and can be represented by a relatively simple function applicable over a range of scales, contaminants, and media.

These problems of dispersion have been receiving considerable attention from chemical, environmental and petroleum engineers, hydrologists, mathematicians and soil scientists. Most of the works reveal common assumption of homogenous porous media with constant porosity, steady seepage flow velocity and constant dispersion coefficient. For such assumption Ebach and White [8] studied the longitudinal dispersion problem for an input concentration that varies periodically with time and Ogata and Banks [16] for a constant input concentration. Hoopes and Herteman [9] studied the problem of dispersion in radial flow from fully penetrating, homogenous, isotropic non adsorbing confined aquifers. Bruce and street [4] considered both longitudinal and lateral dispersion with in semi infinite non adsorbing porous media in a steady unidirectional flow fluid for a constant input concentration. Marino [13] considered the input concentration varying exponentially with time. Al-Niami and Rushton [1] and Marino [12] studied the analysis of flow against dispersion in porous media. Basak [5] presents an analytical solution to the problem of Evaporation from a horizontal soil column in which diffusivity increases linearly with moisture content and also to a problem of concentration dependent diffusion with decreasing concentration at the source. Hunt [10] applied the perturbation method to longitudinal and lateral dispersion in no uniform seepage flow through heterogeneous aquifers. Wang [21] discussed the concentration distribution of a pollutant arising from a instantaneous point source in a two dimensional water channel with non uniform velocity distribution. He employed Gill's method to solve the convective diffusion equation. Kumar

[11] discussed the Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution.

The present paper discusses the analytical solution of the nonlinear differential equation for longitudinal dispersion phenomena which takes places when miscible fluids mix in the direction of flow. The mathematical formulation of the problem yields a non linear partial differential equation. Solution has been obtained by using Bäcklund transformation.

The paper is organised in the following way:

The Bäcklund transformation is introduced in section 1. Formulation of the model and the technique that is applied on dispersion problem to show the efficiency of the proposed approach discussed in section 2. Section 3 ends this works with a brief conclusion.

Bäcklund transformation

Bäcklund transformation and Lax pairs play an important role in solitary theory because the nonlinear iterative principle from Bäcklund transformations converts problem of nonlinear differential equations to purely algebraic equation. Moreover, many connections among Bäcklund transformation, infinite conservation law and inverse scattering, etc. can be found by using Bäcklund transformations [6, 19]. Lax pairs convert nonlinear differential equations to a pair of linear equations.

Cheng et.al [6] defined the Painleve property for partial differential equations. Bäcklund transformation and Lax pairs are obtained by truncating the expansion. Precisely if the singularity manifold is determined by $\phi(x,t) = 0$ and $u(x,t) = 0$, then it is a solution of the partial differential equation

$$u_t = K(u, u_x, ...) \tag{1}$$

Suppose

$$u(x,t) = \frac{1}{\phi^\alpha} \sum_{j=0}^{\infty} u_j(x,t)\phi^j \tag{2}$$

where α is a positive integer, $\phi(x,t)$ and $u_j(x,t)$ are analytic functions in a neighborhood of the manifold $\phi = 0$.

Substituting equation (1.2) into (1), it determines the possible α and the recursion relations for u_j , $j = 0, 1, 2 \ldots$. Bäcklund transformations can be obtained by truncating the expansion.

The main steps of our method are as follows.

Suppose that the solution for differential equation (1) is of the form

$$u = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + u_1 \tag{3}$$

where α is a positive integer u_1 is also a solution of (1) and f is determined later

1. By Substituting (3) into (1), it determines the possible α the highest equality degree in nonlinear terms and a highest order partial derivative terms.
2. By Substituting (3) into (1) collecting all the terms with the highest degree of ϕ_x and setting its coefficient to zero, we obtain an ordinary differential equation Then $f(\phi)$ can be determined.
3. By Collecting all the terms with same order derivatives and setting their coefficients to zero, the compatibility conditions can be obtained.

2 Mathematical formulation and solution of the problem

In many oil reservoirs, once primary oil recovery has ceased, more of the oil may still be recovered by secondary oil recovery. In secondary oil recovery, water is injected into the oil reservoir through one well, displacing the oil so that it can be extracted from a neighbouring well. During second recovery, Consider saturated flow through a porous medium of an oil reservoir and let a portion of the flow domain contain a certain mass of oil. This water will be referred to as a *tracer.* The tracer which is a labelled portion of the same liquid may be identified by its density, colour, electrical conductivity etc. Experience shows that as flow takes place, the tracer gradually spreads and occupies an ever-increasing portion of the flow domain, beyond the region it is expected to occupy according to the average flow alone. This spreading phenomenon is called *hydrodynamic dispersion*

(dispersion, miscible displacement) in a porous medium. It is a non steady, irreversible process (in the sense that the initial tracer distribution cannot be obtained by reversing the flow) in which the tracer mass mixes with the non labeled portion of the liquid. In this work it has considered that the dispersion zone is in one direction i.e X-direction. Parabolic shape that is marked in the Figure [1 and 2] termed as the dispersion flow zone.

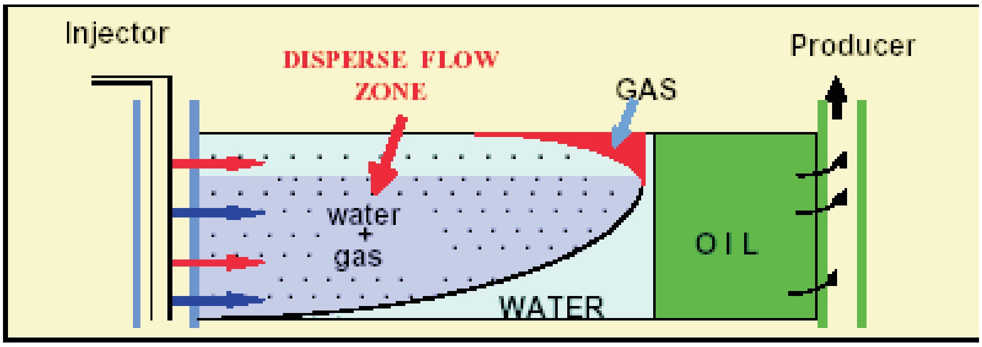

Figure 1: Schematic of the gas-water gravity segregation in far-wellbore region

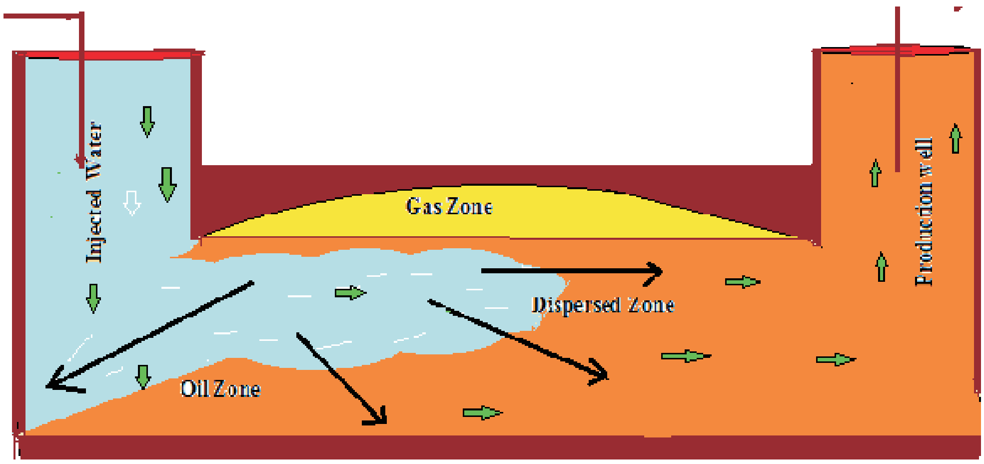

Figure 2: Dispersion zone of water in oil reservoir

The problem is to find the concentration of tracer as a function of time ‘t’ and position ‘x’ as the miscible fluid flow through porous media in an oil reservoir on either sides of the mixed region . The single fluid equation describes the motion of fluid. The problem becomes more complicated in one dimension with fluids of equal properties. Here the mixing takes place longitudinally as well as transversely at $t = 0$ and a dot of fluid having $[C_0]$ concentration is injected over the phase. The dot moves in the direction of flow as well as perpendicular to the flow. Finally it takes the shape of the ellipse with a different concentration $[C_n]$.

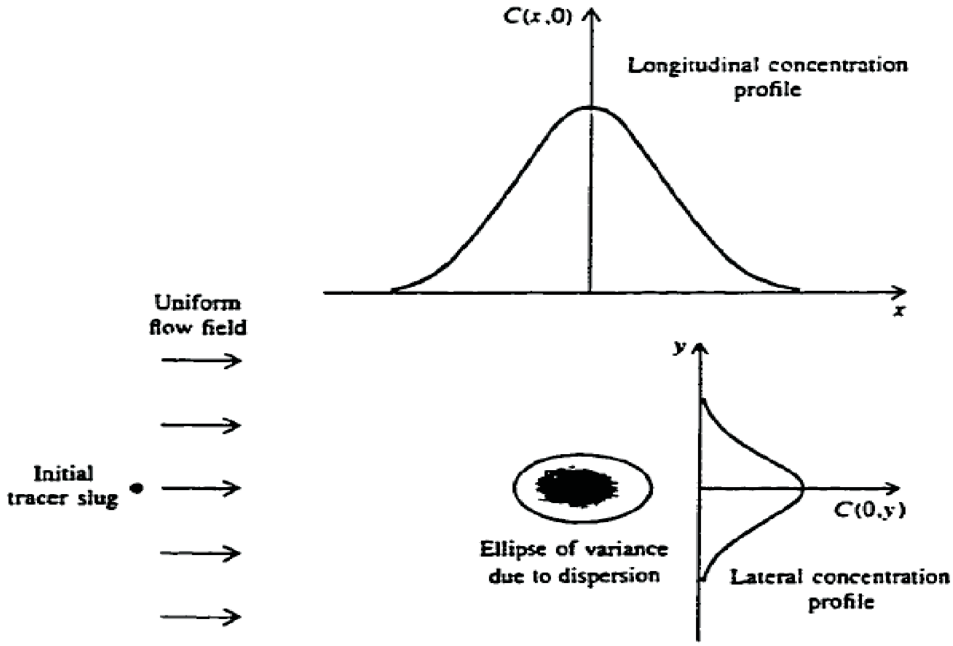

Figure 3: Dispersion of an instantaneous point source in a uniform flow field

In one dimensional analysis of flows in pipes and channels, the mean velocity V over the cross-section (discharge/area) is used to describe the flow. Since the velocity actually varies over the cross-section, two fluid elements at the same cross section at one instant will separate as they travel with different speeds. At the same time, their motion is influenced by molecular and turbulent activities. It has been shown by Taylor [20] and Aris [2] that for sufficiently long dispersion time. In moving through the random passages of the medium, two fluid elements adjacent to each other at one time will separate, as they may take different routes. This geometrical dispersion is coupled with molecular diffusion, turbulent diffusion and dispersion due to non uniformity of velocity across the cross section of the passages. By considering the passage as randomly connected tubes, de Jongs [7] and Saffman [18] have shown that the dispersion in an isotropic medium can be described with a coefficient D_L for longitudinal dispersion in the direction of the seepage velocity and a coefficient D_T for transverse dispersion.

By dimensional analysis, it can be shown that $\frac{D_L}{v}$ and $\frac{D_T}{v}$ (where v is the kinematics viscosity of the fluid) are function of the Reynolds number $\frac{\tilde{V}d}{v}$ ($\tilde{V}$ being the seepage velocity and d the representative grain size of the medium), the Schmidt number $\frac{v}{D_m}$ and the geometrical properties of the medium.

The dispersion equation for a homogenous porous medium without increasing or decreasing the dispersing material is given by (Bear [3])

$$\frac{\partial C}{\partial t} + \nabla . \left(C\tilde{V} \right) = \nabla . \left[\rho \tilde{D} \nabla \left(\frac{C}{\rho} \right) \right] \tag{4}$$

where C be the concentration of the homogenous porous medium. $\tilde{D}$ be the tensor coefficient of dispersion, the non zero components of $\tilde{D}$ are $D_{11} = D_L$ and $D_{22} = D_{33} = D_T$.

In a laminar flow for an Incompressible fluid through homogeneous porous medium, density ρ is constant. Then equation (4) becomes

$$\frac{\partial C}{\partial t} + \tilde{V}.\nabla C = \nabla . \left(\tilde{D} \nabla C \right) \tag{5}$$

Let us assume that the seepage velocity $\tilde{V}$ is along the x- axis, then $\tilde{V} = u(x,t)$ and the non zero components will be $D_{11} \approx D_L = \gamma$ (coefficient of longitudinal dispersion) and other components will be zero [15].

Equation (2.2) becomes

$$d\partial C \partial t + u\frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2} \tag{6}$$

Which describe the convective dispersion in a channel with out increasing or decreasing the dispersing material and u be the flow velocity.

Here u is the component of flow velocity $\tilde{V}$ along x-axis which is time dependent as well as concentration along x axis in $x \geq 0$ direction and $D_L > 0$ and it is the cross sectional flow velocity in porous media.

By Mehta [14], it has been observed that seepage flow velocity u is related with concentration of the dispersing material as

$$u = \frac{C(x,t)}{C_0}, \tag{7}$$

where $x > 0$.

Now for $C_0 \cong 1$ (constant)

Equation (2.3) becomes

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} - \gamma\frac{\partial^2 C}{\partial x^2} = 0 \tag{8}$$

This is the non linear Burger's equation for longitudinal dispersion of miscible fluid flow through porous media.

The theory that follows is confined to dispersion in unidirectional seepage flow through semi-infinite homogeneous porous media. The seepage flow velocity is assumed unsteady. The dispersion systems to be considered are subject to an input concentration of contaminants C_0. The porous medium is considered as nonadsorbing. To obtain the solution to an unsteady flow problem a new approach to Bäcklund transformations of nonlinear evolution equation is presented.

Consider the input concentration is C_0. The governing partial differential equation (8) for longitudinal hydrodynamic dispersion with in a semi-infinite nonadsorbing porous medium in a unidirectional flow field in which D is the longitudinal dispersion coefficient, C is the average cross-sectional concentration, u is the unsteady seepage velocity, **x** is a coordinate parallel to flow and t is time.

The initial and boundary conditions are

$$\left.\begin{array}{l} C(0,t) = C_0, t \geq 0 \\ C(l,t) = C_1, t \geq 0 \end{array}\right\} \quad \text{Provided} \quad C_1 \prec C_0 \tag{9}$$

Suppose that its solution of (8) is of the form

$$C = \frac{\partial^\alpha}{\partial x^\alpha} f(\phi) + C_1(x,t) \tag{10}$$

Requiring the equality highest degree of ϕ_x in CC_x and $-C_{xx}$, we find $\alpha = 1$, there fore

$$C = f'(\phi)\phi_x + C_1(x,t) \tag{11}$$

Substituting (11) into (10) and calculating, we have

$$\begin{aligned}
&C_t + CC_x - \gamma C_{xx} = (f'f'' - \gamma f''')\phi_x^3 + \\
&(f''\phi_t\phi_x + f'^2\phi_x\phi_{xx} + C_1 f''\phi_x^2 - 3\gamma f''\phi_x\phi_{xx}) \\
&+(\phi_{tx} + C_{1x}\phi_x + C_1\phi_{xx} - \gamma\phi_{xxx})f' + \\
&(C_{1t} + C_1 C_{1x} - \gamma C_{1xx}) = 0
\end{aligned} \tag{12}$$

Setting

$$f'f'' - \gamma f''' = 0 \tag{13}$$

Which has a solution

$$f = -2\gamma\log\phi \tag{14}$$

There by

$$f'^2 = 2\gamma f'' \tag{15}$$

Substituting (2.11) into (12) and using (2.10) we obtain

$$\begin{aligned}
&C_t + CC_x - C_{xx} = (\phi_t\phi_x + C_1\phi_x^2 - \gamma\phi_x\phi_{xx})f'' + \\
&\tfrac{\partial}{\partial x}(\phi_t + C_1\phi_x - \gamma\phi_{xx})f' + C_{1t} + C_1 C_{1x} - \gamma C_{1xx} = 0
\end{aligned}$$

Setting the coefficient of f'', f''' and final linear combination term of C_1 to zero gives

$$\phi_x(\phi_t + C_1\phi_x - \gamma\phi_{xx}) = 0 \tag{16}$$

$$\frac{\partial}{\partial x}(\phi_t + C_1\phi_x - \gamma\phi_{xx}) = 0 \tag{17}$$

$$C_{1t} + C_1 C_{1x} - \gamma C_{1xx} = 0 \tag{18}$$

$\phi_x \neq 0$ implies from (16), $\phi_t + C_1\phi_x - \gamma\phi_{xx} = 0$

Compatibility condition (17) is satisfied. Substituting (2.11) in to (11), we obtain Bäcklund transformation

$$C = -2\gamma \frac{\phi_x}{\phi} + C_1 \tag{19}$$

where ϕ satisfies (2.16) and C_1 satisfies (18).

If $C_1 = 0$, we obtain Cole-Hopf transformation

$$C = -2\gamma \frac{\phi_x}{\phi} \tag{20}$$

Equation (2.17) reduces (8) in to diffusion equation.

$$\frac{\partial \phi}{\partial t} = \gamma \frac{\partial^2 \phi}{\partial x^2} \tag{21}$$

Let $\phi = \phi(x,t)$ be a solution of homogenous parabolic diffusion equation (2.18) with nonzero initial condition

$$\phi(0,t) = \frac{2C_0^*}{C_0} \quad \phi(l,t) = \frac{2C_1^*}{C_1} \tag{22}$$

By using similarity transformation $\phi = g(\eta)$ and $\eta = \dfrac{x}{\sqrt{4\gamma t}}$

Equation (2.18) becomes

$$-\frac{dg}{d\eta}\eta = \frac{1}{2}\frac{d^2g}{d\eta^2}$$

The solution of (2.18) is

$$g(\eta) = \frac{2C_0^*}{C_0} + k\int_0^{\eta} e^{-\eta^2} d\eta$$

with initial condition

$$g(0) = \frac{2C_0^*}{C_0} \tag{23}$$

Using condition (2.20) we get

$$k = \frac{2\left(\dfrac{C_1^*}{C_1} - \dfrac{C_0^*}{C_0}\right)}{\int\limits_0^l e^{-\eta^2} d\eta}$$

Therefore we get

$$\phi = 2\left(\frac{C_0^*}{C_0} + \frac{\left(dC_1^* C_1 - \dfrac{C_0^*}{C_0}\right)\int\limits_0^\eta e^{-\eta^2} d\eta}{\int\limits_0^l e^{-\eta^2} d\eta}\right) \tag{24}$$

where $\eta = \dfrac{x}{\sqrt{4\gamma t}}$

Equation (2.21) and (8) gives

$$C(x,T) = \frac{\dfrac{\partial}{\partial x}\left[2\left(\dfrac{C_0^*}{C_0} + \dfrac{\left(\dfrac{C_1^*}{C_1} - \dfrac{C_0^*}{C_0}\right)\int\limits_0^\eta e^{-\eta^2} d\eta}{\int\limits_0^l e^{-\eta^2} d\eta}\right)\right]}{\left(\dfrac{C_0^*}{C_0} + \dfrac{\left(\dfrac{C_1^*}{C_1} - \dfrac{C_0^*}{C_0}\right)\int\limits_0^\eta e^{-\eta^2} d\eta}{\int\limits_0^l e^{-\eta^2} d\eta}\right)} \tag{25}$$

where $\eta = \dfrac{x}{\sqrt{4\gamma t}}$ is the concentration C of porous medium.

3 Conclusions

Mathematical formulation has been developed for predicting the possible concentration of a given dissolved substance in unsteady unidirectional seepage flow through semi-infinite, homogeneous, isotropic porous media subject to the source of concentrations that vary exponentially with time.

The expression taken into account is the mass transfer from liquid matrix to solid matrix due to adsorption. The analytical solution obtained here in equation (2.22) represents concentration of the dispersing material in a porous medium which is useful to the study of salinity intrusion in groundwater, helpful in making quantitative predictions on the possible contamination of groundwater supplies resulting from groundwater movement through buried wastes and concentration of dispersed fluid in oil reservoir during secondary recovery process.

References

[1] Al-Niami., , A. N. S and Rushton, K. R. (1977): " Analysis of Flow against dispersion in porous media", *J. Hydrol.*, Vol. 33, pp 87-97.

[2] Aris, R. (1956): "On the Dispersion of a solute in a fluid flowing through a tube", Proc.Roy.Soc.(London),A235,Apr..

[3] Bear, J. (1972): Dynamics of fluids in porous media, American Elsevier; New York.

[4] Bruce, J. C., and Street, R. L. (1966) "Studies of free surface flow and two dimensional dispersion in porous media", Report No. 63, Civil Eng. Dept. Stanford Univ., Stanford, California.

[5] Basak, P. (1978) :"Evaporation from horizontal soil columns with variable diffusivity", *J. Hydrology.*, Vol. 39, pp 129-136.

[6] Cheng, Y. et al, (1991): "Connections among symmetries, Bäcklund transformation and Painleve property", *Acta. Math. Appl. Sinica*, Vol.14, pp 180–184. (in Chinese)

[7] De Jong, G.(1958): "Longitudinal and Transverse Diffusion in Granular Deposits", Trans.Amer.Geophys.Union, 39, No.1.

[8] Ebach, E. H., and White, R. (1958): "Mixing of fluids flowing through beds of packed solids", A. *J. Ch. E.*, Vol. 4, p. 161.

[9] Hoopes, J. A, and Harteman, D. R. F. (1965): "Waste water recharge and dispersion in porous media", Technical Report No. 75.

[10] Hunt, B. (1978): "Dispersion calculations in nonuniform seepage, *J. Hydrol.*, Vol.36, 261-277"

[11] Kumar, Naveen (1983): " Dispersion of Pollutants in Semi-Infinite Porous Media with Unsteady Velocity Distribution", *Nordic Hydrology,* 1983, pp 167-178

[12] Marino, M. A. (1974): "Distribution of contaminants in porous media flow", Water Resource, Vol. 10, No. 5.

[13] Marino, M. A. (1978) "Flow against dispersion in non adsorbing porous media"*, J. Hydrol.*, Vol. 37, pp 149-158.

[14] Mehta, M. N. (1978): "An asymptotic expansion in fluid flow through porous media; Ph.D. Thesis; S.G. University", Surat (India).

[15] Mehta, M.N., Patel, T. (2006): "A solution of Burger's equation type one dimensional Ground water Recharge by spreading in Porous Media*". Journal of Indian Acad.Math.*Vol.28, No.1(2006) pp 25-32.

[16] Ogata, A., and Banks, R. B.(1961): "A solution of differential equation of longitudinal dispersion in porous media". Profess. Paper No. 411-A.

[17] Polubarinova-Kochina, P. Ya (1962): "Theory of Ground –water Movement", Princeton Uni. Press.

[18] Saffman,P.G.(1959):"A Theory of Dispersion in a porous medium" J.Fluid Mechanics,6,Pt.3,Oct.

[19] Tu, G.Z. (1981): "Bäcklund transformation and conservation law", Acta. Math. Appl.Sinica, 63-68. (in Chinese)

[20] Taylor, G.I. (1953): "Dispersion of soluble matter in solvent flowing slowly through a tube", Proc.Roy.society (London),A219,Aug..

[21] Wang, St. T., McMillan, A. F., and Chen, B. H. (1978): "Dispersion of pollutants in channels with nonuniform velocity distribution", *Water Research*, Vol. 12, pp 389-394

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Novi pristup Beklundovim transformacijama za uzdužnu disperziju tečenja fluida sa mešanjem kroz poroznu sredinu u uljnom rezervoaru pri sekundarnom procesu oporavka

Razvijen je teorijski model za problem disperzije u poroznoj sredini gde je tečenje jednodimenzionalno sa nestacionarnim srednjim tečenjem. Rešenje problema disperzije je prikazano pomoću novog pristupa Beklundovim transformacijama nelinearnim parcijalnim diferencijalnim jednačinama.

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