Theoret. Appl. Mech., Vol.38, No.1, pp. 17–36, Belgrade 2011

On hydromagnetic thermosolutal convection coupled with cross-diffusion in completely confined fluids

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Abstract

The instability of thermosolutal convection coupled with cross-diffusion of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid wall in the presences of a uniform magnetic field applied in an arbitrary direction is investigated. Some general qualitatively results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude are derived. The results for the thermosolutal convection problems with or without the individual consideration of Dufour and Soret effects follow as a consequence.

Keywords: Thermosolutal convection, Dufour effect, Soret effect, Rayleigh numbers, Chandrasekhar number, Prandtl numbers and Lewis number

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Nomenclature

p	growth rate, $[1/s]$
$\vec{q}$	velocity, $[m/s]$
σ	Prandtl number, $[-]$
P	Pressure, $[Pa]$
R_T	Thermal Rayleigh number, $[-]$
R_S	Solutal Rayleigh number, $[-]$
Q	Chandrasekhar number, $[-]$
$\vec{h}$	Magnetic field, [Gauss]
D_T	Dufour number, $[-]$
S_T	Soret number, $[-]$
τ	Lewis number, $[-]$
σ_1	Magnetic Prandtl number [-]
g	acceleration due to gravity, $[m/s^2]$
d	depth of layer, $[m]$
t	time, $[s]$

*G*reek symbols

α	coefficient of thermal expansion, $[1/K]$
α'	coefficient of solute expansion
β_1	uniform temperature gradient, $[K/m]$
β_2	uniform concentration gradient, $[K/m]$
η	electrical resistivity, $[m^2/s]$
κ	thermal diffusivity, $[m^2/s]$
η_1	mass diffusivity,
ν	kinematic viscosity, $[m^2/s]$
ρ	density, $[kg/m^3]$
θ	perturbation in temperature, $[K]$
ϕ	perturbation in concentration, $[Kg]$
λ	the ratio of two magnetic Prandtl numbers [-]
D_T	dimensionless Dufour number, $[-]$
S_T	dimensionless Soret number, $[-]$

1 Introduction

Thermosolutal convection or more generally double-diffusive convection, like its classical counterpart, namely, single –diffusive convection, has carved a niche for itself in the domain of hydrodynamic stability on account of its interesting complexities as a double- diffusive phenomenon as well as its direct relevance in the fields of Oceanography, Asrophysics, Geophysics, Limnology and Chemical engineering etc. can be seen from the review articles by Turner [1] and Brandt and Fernando [2]. An interesting early experimental study is that of Caldwell [3]. The problem is more complex than that of a single - diffusive fluid because the gradient in the relative concentration of two components can contribute to a density gradient just as effectively as can a temperature gradient. Further, the presence of two diffusive modes allows either stationary or overstable flow states at the onset of convection depending on the magnitude of the fluid parameters, the boundary conditions and the competition between thermal expansion and the thermal diffusion. More complicated double- diffusive phenomenon appears if the destabilizing thermal/concentration gradient is opposed by the effect of a magnetic field or rotation.

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour [4, 5] effects. An externally imposed temperature gradient produces a chemical potential gradient and the phenomenon known as the Soret effect, arises when the mass flux contains a term that depends upon the temperature gradient. The analogous effect that arises from a concentration gradient dependent term in the heat flux is called the Dufour effect. The coupling of the fluxes of the stratifying agents is a prevalent feature in multicomponent fluid systems. In general, the stability of such systems is also affected by the cross-diffusion terms. Hurle and Jakeman [6] have studied the effect of Soret coefficient on the double–diffusive convection. They have reported that the magnitude and sign of the Soret coefficient were changed by varying the composition of the mixture. McDougall [7] has made an in depth study of double diffusive convection where in both Soret and Dufour effects are important.

Mohan [8,9] has mollified the nastily behaving governing equations of Dufour- driven thermosolutal convection and Soret – driven thermosolutal convection problems of the Veronis [10] type by the construction of a linear transformation and derived the desired results concerning the linear growth rate and the behavior of oscillatory motions on the lines suggested by Banerjee et. al. [11, 12]. Almost all the papers that are written on the subject are confined to horizontal layer geometry on account of complexity of the problem for arbitrary geometry. However, there do exist a class of results in the domain of hydrodynamic and hydromagnetic stability theory that sparks of their generalization to containers of arbitrary shape [13].

Motivated by these considerations, the present paper investigates the instability of thermosolutal convection coupled with cross-diffusion of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid wall in the presences of a uniform magnetic field applied in an arbitrary direction and derives some general qualitatively results concerning the character of marginal state, stability of oscillatory motions and limitations on the oscillatory motions of growing amplitude. The results for the thermosolutal convection problems with or without the individual consideration of Dufour and Soret effects follow as a consequence.

2 Mathematical formulation and analysis

The relevant governing non-dimensional linearized perturbation equations in the present case with time dependence of the form exp (pt) $(p = p_r + ip_i)$ are given by:

$$\frac{p}{\sigma}\vec{q} = -\nabla(P) - curl\ curl\ \vec{q} + R_T\theta\ \hat{k} - R_S\phi\ \hat{k} + Q\left(curl\vec{h}\right) \times \hat{l} \quad (1)$$

$$\left(\nabla^2 - p\right)\theta + D_T\nabla^2\phi = -\vec{q}\cdot\hat{k} \quad (2)$$

$$\left(\tau\nabla^2 - p\right)\phi + S_T\nabla^2\theta = -\vec{q}\cdot\hat{k} \quad (3)$$

$$curlcurl\vec{h} + \frac{p\sigma_1\vec{h}}{\sigma} = curl\left(\vec{q}\times\hat{\ell}\right) \quad (4)$$

and

$$\nabla \cdot \vec{q} = 0 = \nabla \cdot \vec{h} \tag{5}$$

In the above equations $\vec{q}\,(x,y,z)\,,\ P\,(x,y,z)\,,\ \theta\,(x,y,z)\,,\ \phi\,(x,y,z)$ and $\vec{h}\,(x,y,z)$ respectively denote the perturbed velocity, pressure, temperature, concentration and magnetic field and are complex valued functions defined on V, $R_T = \dfrac{g\alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $R_S = \dfrac{g\alpha'\beta' d^4}{\kappa'\upsilon}$ is the concentration Rayleigh number, $Q = \dfrac{\mu_e H_0^2 d^2}{4\pi\rho_0\upsilon\eta}$ is the Chandrasekhar number, $\sigma = \dfrac{\upsilon}{\kappa}$ is the Prandtl number, $\tau = \dfrac{\eta_1}{\kappa}$ is the Lewis number, $D_T = \dfrac{\beta_2 D_f}{\beta_1\kappa}$ is the Dufour number, $S_T = \dfrac{\beta_1 S_f}{\beta_2\eta}$ is the Soret number, and $\hat{k}$ is a unit vertical vector. Further, with d as the characteristic length, the equations have been cast into dimensionless forms by using the scale factors $\dfrac{\kappa}{d}$, $\dfrac{d^2}{\kappa}$, βd, $\dfrac{\rho\upsilon\kappa}{d^2}$, $\beta' d$ and $\dfrac{\kappa H_0}{\eta}$ for velocity, time, temperature, pressure, concentration and magnetic field respectively.

We seek solutions of equations (1)-(5) in the simply connected subset V of R^3 subject to the following boundary conditions:

either

$$\vec{q} = 0 = \theta = \phi = \hat{n} \times curl\,\vec{h} \quad on \quad S \tag{6}$$

or

$$\vec{q} = 0 = \nabla\theta \cdot \hat{n} = \nabla\phi.\, \hat{n} = \hat{n} \times curl\,\vec{h} \quad on \quad S \tag{7}$$

where $\hat{n}$ is a unit vector in the direction of the normal to boundary surface S.

We now prove the following lemmas and theorems:

Lemma 1: (Poincare's Inequality) – If f (x, y, z) is any smooth function which vanishes on S and ℓ is the smallest distance between two parallel planes which just contains V, then there exists a constant $\lambda_0 (> 2)$ such that

$$\int_V |\nabla f|^2\, dv \geq \frac{\lambda_0}{\ell^2} \int |f|^2\, dv \tag{8}$$

Proof: Joseph [14].

Lemma 2: If $\left(p, \vec{q}, \vec{h}, \theta, \phi\right)$ is a non-trivial solution of equation (1)-(5) together with either of the boundary conditions(6)-(7), then the following integral relations hold :

$$\int_{V} \vec{q}^{*} \cdot curl\ curl\, \vec{q} dv = \int_{V} |curl\ \vec{q}|^{2} dv, \tag{9}$$

$$\int_{V} \vec{q}^{*} \cdot curl\ curl\ \left(\vec{q} \times \hat{\ell}\right) dv = \int_{V} curl\ \left(\vec{q} \times \hat{\ell}\right) \cdot curl\, \vec{q}^{*} dv, \tag{10}$$

$$\int_{V} \vec{q}^{*} \cdot curl\ curl\ \left(\theta\ \hat{k}\right) dv = 0 = \int_{V} \vec{q}^{*} \cdot curl\ curl\ \left(\phi\ \hat{k}\right) dv, \tag{11}$$

$$\int_{V} \vec{q}^{*} \cdot \left[\left(curl\, \vec{h}\right) \times \hat{\ell}\right] dv = -\int_{V} \vec{h} \cdot curl\ curl\ \left(\vec{q}^{*} \hat{\ell}\right) dv, \tag{12}$$

$$\int_{V} \vec{q}^{*} \cdot \left[\hat{\ell}\ curl\ curl\ curl\ \vec{h}\right] dv = -\int_{V} curl\ curl\ \vec{h} \cdot\ \ curl \left(\vec{q}^{*} \hat{\ell}\right) dv, \tag{13}$$

$$\int_{V} \vec{h}^{*} \cdot\ curl\ curl\ curl\ \vec{h}\, dv = -\int_{V} \left|curl\ \vec{h}\right|^{2} dv =$$

$$\int_{V} \vec{h}^{*} \cdot\ curl\ curl\ curl\ \vec{h}^{*}\, dv, \tag{14}$$

$$\int_{V} \vec{q}^{*} \cdot \nabla \left(P\right) dv = 0, \tag{15}$$

$$\int_{V} \vec{q}^{*} \cdot \left[\nabla\ \left(div\, \theta\ \hat{k}\right)\right] dv = 0 = \int_{V} \vec{q}^{*} \cdot \left[\nabla\ \left(\phi\ \hat{k}\right)\right] dv, \tag{16}$$

$$\int_{V} \vec{q}^{*} \cdot \left[\nabla\ \left(\hat{\ell} \cdot curl\ curl\ \vec{h}\right)\right] dv = 0, \tag{17}$$

$$\int\limits_{V} \theta^* \nabla^2 \theta dv = -\int\limits_{V} |\nabla \theta|^2 \, dv = \int \theta^* \nabla \theta^* dv, \tag{18}$$

and

$$\int\limits_{V} \phi^* \nabla^2 \phi dv = -\int |\nabla \phi|^2 \, dv = \int \phi^* \nabla \phi^* dv \tag{19}$$

where '*' denotes complex conjugate and $\left|\vec{A}\right|^2 = \vec{A} \cdot \vec{A}^*$ for any vector $\vec{A}$.

Proof: If $\vec{A}$, $\vec{B}$ *and* $\vec{C}$ are smooth vector-valued functions and ψ is a smooth scalar-valued function on V such that $\vec{A} \times \vec{B}$ *and* $\Psi\vec{C}$ vanish on S, then using Gauss' divergence theorem and the vector identities

$$div\left(\vec{A} \times \vec{B}\right) = \vec{B}.curl\ \vec{A} - \vec{A} \cdot curl\vec{B}$$

and $div\left(\Psi\vec{C}\right) = \nabla\Psi.\vec{C} + \Psi\ div\ \vec{C}$ it follows that

$$\int\limits_{V} \vec{B} \cdot curl\ \vec{A}\ dv = \int\limits_{V} \vec{A} \cdot curl\ \vec{B}\ dv, \tag{20}$$

and

$$\int\limits_{V} \nabla\Psi \cdot \vec{C}\ dv = -\int\limits_{V} \Psi\ div\ \vec{C}\ dv \quad . \tag{21}$$

Now integral relations (9)–(14) follow from equation (20) by choosing $\vec{A}$ *and* $\vec{B}$ appropriately and integral relations (15)-(19) follow from equation (21) by choosing Ψ *and* $\vec{C}$ appropriately. This completes the proof of the lemma.

Theorem 1: If $\left(p, \vec{q}, \vec{h}, \theta, \phi\right)$, $p = p_r + ip_i$ is a non-trivial solution of equations(1) - (5) together with either of the boundary conditions (6) - (7), $R_T' > 0,\ R_S' > 0$ and $\frac{\tau\ k_2}{k_1} R_T' \leq R_S'$ then $p_r = 0 \Rightarrow p_i \quad \neq 0$.

Proof: We introduce the transformations

$$\breve{\vec{q}} = \left(S_T + B\right) \vec{q}, \quad \tilde{\theta} = E\theta + F\phi,$$

$$\tilde{\phi} = S_T\theta + B\phi, \quad \tilde{h}_z = (\ S_T + B)\ h_z \tag{22}$$

where

$$B = -\frac{1}{\tau}A, \quad E = \frac{S_T + B}{D_T + A}A, \quad F = \frac{S_T + B}{D_T + A}D_T$$

and A is a positive root of the equation

$$A^2 + (\tau - 1)A \ - \tau\ S_T D_T = 0.$$

The systems of equations (1)-(5), upon using the transformations (22) assume the following forms:

$$\frac{p}{\sigma}\vec{q} = -\nabla P - curl\ curl\ \vec{q} + R'_T\theta\ \hat{k} - R'_S\phi\ \hat{k} + Q\left(curl\ \ \vec{h}\right) \times \hat{\ell}, \tag{23}$$

$$(\ k_1\nabla^2 - p\ )\ \theta = -\vec{q}\cdot\hat{k}, \tag{24}$$

$$(\ \ k_2\tau\nabla^2 - p\ )\ \phi = -\vec{q}\cdot\hat{k}, \tag{25}$$

$$curlcurl\vec{h} + \frac{p\sigma_1\vec{h}}{\sigma} = curl\left(\vec{q}\times\hat{\ell}\right) \tag{26}$$

$$\nabla\cdot\vec{q} = 0 = \nabla\cdot\vec{h} \tag{27}$$

with

$$\vec{q} = 0 = \theta = \phi = \hat{n}\times curl\ \vec{h}\ \ on\ \ S \tag{28}$$

or

$$\vec{q} = 0 = \nabla\theta\cdot\hat{n} = \nabla\phi\cdot\hat{n} = \hat{n}\times curl\ \vec{h}\ \ on\ \ S, \tag{29}$$

where

$k_1 = 1 + \dfrac{\tau\ D_T S_T}{A},\ k_2 = 1 - \dfrac{S_T D_T}{A}$ *are positive constants*

and $R'_T = \dfrac{(D_T + A)(R_T B + R_S S_T)}{BA - S_T D_T},\ R'_S = \dfrac{(S_T + B)(R_S A + R_T D_{T)}}{BA - S_T D_T}$

are respectively the modified thermal Rayleigh number and the modified concentration Rayleigh number.

The sign tilde has been omitted for simplicity.

Suppose $p_r = 0 \Rightarrow p_i = 0$. Then, $p = 0$ and equations (23) – (26) become

$$\nabla P + curl\ curl\ \vec{q} = R'_T \theta\ \hat{k} - R'_S \phi\ \hat{k} + Q\left(curl\ \ \vec{h}\right) \times \hat{\ell}, \tag{30}$$

$$k_1 \nabla^2\ \theta = -\vec{q} \cdot \hat{k}, \tag{31}$$

$$k_2 \tau \nabla^2\ \phi = -\vec{q} \cdot \hat{k}, \tag{32}$$

$$curl curl \vec{h} = curl\left(\vec{q} \times \hat{\ell}\right) \tag{33}$$

If $\zeta = k_1 \theta - \tau\ k_2 \phi$, then it follows from equations (31)-(32) that

$$\nabla^2 \zeta = 0 \tag{34}$$

Further, in view of boundary conditions (28)-(29), we have either

$$\zeta = 0\ \ or\ \ \nabla \zeta \cdot \hat{n} = 0\ \ on\ \ S \tag{35}$$

The only solution of equation (34) in V subject to either of the boundary condition in (35) is ζ= 0. Consequently equation (30) assume the form

$$\nabla P + curl\ curl\ \vec{q} = \left(\frac{\tau\ k_2}{k_1} R'_T - R'_S\right) \phi\ \hat{k} + Q\left(curl\ \vec{h}\right) \times \hat{\ell} \tag{36}$$

Taking dot product of equation (36) with $\vec{q}^*$, integrating the resulting equation over the domain V and using lemma 2, we get

$$\int\limits_V |curl\ \vec{q}|^2\, dv + Q \int \vec{h} \cdot curl\left(\vec{q} \times \hat{\ell}\right) dv =$$

$$\left(\frac{\tau\ k_2}{k_1} R'_T - R'_S\right) \int\limits_V \phi\left(\vec{q}^* \cdot \hat{k}\right)\, dv \tag{37}$$

Equation (37) upon using equation (32) and (33) and then appealing to lemma 2 yields the equation

$$\int\limits_V |curl\ \vec{q}|^2\, dv + Q \int\limits_V \left|curl\ \vec{h}\right|^2 = k_2 \tau \left(\frac{\tau\ k_2}{k_1} R'_T - R'_S\right) \int\limits_V |\nabla \phi|^2 dv. \tag{38}$$

It follows from equation (38) that $\frac{\tau\ k_2}{k_1} R'_T > R'_S$, a result contrary to the given hypothesis of the theorem. Hence $p_r = 0 \Rightarrow p_i \quad \neq 0$.

This completes the proof of the theorem.

Theorem 1, in the parlance of linear stability theory, may be stated as follows:

PES is not valid for the hydromagnetic thermosolutal convection coupled with cross-diffusion if $\frac{\tau\ k_2}{k_1} R'_T \leq R'_S$.

Cor.1. PES is not valid for thermosolutal convection coupled with cross-diffusion if $\tau\ R'_T \leq R'_S$

Cor.2. PES is not valid for Dufour-driven hydromagnetic thermosolutal convection ($S_T = 0,\ k_1 = k_2 = 1$) if $\tau\ \left(R_T + \frac{R_T D_T}{1-\tau}\right) \leq (\ R_S + \frac{R_T D_T}{1-\tau}\ )$.

Cor.3. PES is not valid for Soret-driven hydromagnetic thermosolutal convection ($D_T = 0, k_1 = k_2 = 1$) if $\tau\ \left(R_T - \frac{\tau R_S S_T}{1-\tau}\right) \leq (\ R_S - \frac{\tau\ R_S S_T}{1-\tau}\ )$.

Theorem 2: If $\left(p, \vec{q}, \vec{h}, \theta, \phi\right)$, $p = p_r + ip_i$ is a non-trivial solution of equations (23) - (27) together with either of the boundary conditions (28) - (29),$R'_T > 0, R'_S > 0,\ \ k_2 < k_1$, and $R'_T \leq R'_S$ and $\tau = 1$ then $p_r < 0$

Proof: Since $\tau = 1$, therefore it follows from equations (24) and (25) and boundary conditions (28)-(29) that

$$(\langle k_1 - k_2 \rangle\ \nabla^2 - p\ )\ \chi = 0, \tag{39}$$

where

$$\chi = (\theta - \phi) \quad \textit{and either} \quad \chi = 0 \ \textit{or} \ \ \nabla\chi.\hat{n} = 0 \ \textit{on}\ S \tag{40}$$

Multiplying equation (39) by $\chi*$, integrating over the domain V, using equation (21) with $\varphi = \chi*$ and $\vec{q} = \nabla\chi$ and equating the real part of the resulting equation, we get

$$(k_1 - k_2) \int_v |\nabla\chi|^2 dv + p_r \int_v |\chi|^2 dv = 0 \tag{41}$$

Suppose $p_r \geq 0$. Then it follows from equation (41) that $\chi = 0$

Consequently, taking the dot product of equation (23) with $\vec{q}*$, integrating the resulting equation over the domain V and invoking lemma 2 and equations (25) - (26), we get

$$\begin{aligned} &\frac{p}{q}\int_v |\vec{q}|^2 dv + \int_v |curl\vec{q}|^2 dv = (R'_T - R'_S)\int_v k_2 |\nabla\phi|^2 + \\ &p*|\phi|^2 dv - Q\int_v \left|curl\ \vec{h}\right|^2 dv - \frac{Q\sigma_1}{\sigma} p*\int_v \left|\vec{h}\right|^2 dv. \end{aligned} \tag{42}$$

Equating real parts of equation (42), we have

$$\begin{aligned} &\frac{p_r}{q}\int_v |\vec{q}|^2 dv + \int_v |curl\vec{q}|^2 dv + Q\int_v \left|curl\ \vec{h}\right|^2 dv + \frac{Q\sigma_1}{\sigma} p_r \int_v \left|\vec{h}\right|^2 dv \\ &\qquad = \ .(R'_T - R'_S)\int_v k_2 |\nabla\phi|^2 + p_r |\phi|^2 dv \end{aligned} \tag{43}$$

It follows from equation (43) that $R'_T > R'_S$, a result contrary to the given hypothesis of the theorem.

Hence, we must have $p_r < 0$. This completes the proof of the theorem.

Theorem 2 implies that hydromagnetic thermosolutal convection coupled with cross-diffusion is stable if the Lewis number $\tau = 1$

Cor.4. The thermosolutal convection coupled with cross-diffusion is stable if the Lewis number $\tau = 1$.

Cor.5. An initially bottom heavy $(R_T < R_S)$ thermosolutal convection of the Veronis type $(R_T > 0, R_S > 0)$ is stable if $\tau = 1$.

Theorem 3: If $\left(p, \vec{q}, \vec{h}, \theta, \phi\right)$, $p = p_r + ip_i$ is a non-trivial solution of equations (23) - (27) together with either of the boundary conditions (28) - (29),$R'_T > 0, R'_S > 0$, and $k_1 \le \lambda \le \tau\, k_2$, then for large Q (or for large $R_S\; if\; Q = 0$)

$$p_r \ge 0 \Rightarrow p_i = 0$$

where

$$\lambda = \begin{cases} \tau,\; if\; R_S > 0\; and\; Q = 0 \\ \dfrac{\sigma}{\sigma_1},\; if\; R_S \ge 0\, and\, Q > 0. \end{cases}$$

Proof: Operating on equation (23) by $(\lambda\, curl\, curl\; + p)$ and using the vector identities

$$curl\;\left(\Psi\vec{A}\right) = \Psi curl\,\vec{A} + \nabla\Psi + \vec{A}$$

$$curl\;\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{A} - \left(\vec{A}\cdot\nabla\right)\vec{B} + \vec{A}\; div\;\vec{B}$$

and

$$\nabla\;\left(\vec{A}\times\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{A} + \left(\vec{A}\cdot\nabla\right)\vec{B} + \;\vec{B}\; curl\;\vec{A} + \vec{A}\times curl\;\vec{B},$$

with an appropriate choice of $\Psi, \vec{A}$ and $\vec{B}$, it follows that

$$\begin{aligned} & p\left(1+\frac{\lambda}{\sigma}\right) curl\; curl\;\vec{q} + \frac{p^2}{\sigma}\vec{q} + p\nabla P + \\ & R'_S\left\{\lambda\left[\nabla\left(div\;\phi\hat{k}\right) - \nabla^2\phi\,\hat{k}\right] + p\phi\,\hat{k}\right\} \\ & -R'_T\left\{\lambda\left[\nabla\left(div\;\theta\,\hat{k}\right) - \nabla^2\theta\,\hat{k}\right]\right\} + \\ & Q\left\{\lambda\left[\hat{\ell}\times curl\; curl\; curl\;\vec{h} - \nabla\left(\hat{\ell}\times curl\; curl\;\vec{h}\right)\right.\right. \\ & \left.\left.-p\left(curl\;\vec{h}\right)\times\hat{\ell}\right\}\right] = -\lambda\, curl\; curl\; curl\; curl\;\vec{q} \end{aligned} \tag{44}$$

Taking the dot product of equation (44) with $\vec{q}^{*}$, integrating the resulting equation over the domain V and using lemma 2, we have

$$p\left(1+\frac{\lambda}{\sigma}\right)\int\limits_{V}|curl\;\vec{q}|^2\; + \frac{p^2}{\sigma}\int\limits_{V}|\vec{q}|^2 dv + R'_T\int\limits_{V}\left(\lambda\nabla^2\theta - p\theta\right)\left(\vec{q}^{*}\cdot\hat{k}\right)dv$$

$$-R'_S \int\limits_V \left(\lambda \nabla^2 \phi - p\phi\right) \left(\vec{q}^* \cdot \hat{k}\right) dv + pQ \int\limits_V \vec{h}\ curl \left(\vec{q} \cdot \hat{\ell}\right) dv$$

$$+Q\lambda \int\limits_V curl\ \ curl\ \vec{h}\ curl \left(\vec{q} \cdot \hat{\ell}\right) dv$$

$$= -\ \lambda \int\limits_V \vec{q}\ curl\ \ curl\ \ curl\ \ curl\ \ \vec{q}\ dv \tag{45}$$

Since Q (the ratio of magnetic to viscous forces) is very large, the effect of viscosity is thus significant near the bounding surfaces and in the above equation the integral on the right hand side (resulting from the viscous forces) is negligible in comparison with the last integral on the left hand side (resulting from the magnetic force) (c.f. Sherman and Ostrach). Consequently, taking the right hand side of equation (45) to zero, eliminating $\left(\vec{q}^*.\hat{k}\right)$ *and* $\left(\vec{q}^* \times \hat{\ell}\right)$ from the resulting equation by using equation (24)-(26) and then appealing to lemma 2, we get

$$p\left(1 + \frac{\lambda}{\sigma}\right) \int\limits_V |curl\ \vec{q}|^2 dv\ + \frac{p^2}{\sigma} \int\limits_V |\vec{q}|^2 dv -$$

$$R'_T \int\limits_V \left(\left(k_1 \lambda \left|\nabla^2 \theta\right| + |p|^2 |\theta|^2\right) dv - R'_T \left(p^* \lambda + p k_1\right)\right.$$

$$\int\limits_V |\nabla \theta|^2 dv + R'_S \int\limits_V \left(\lambda k_2 \left|\nabla^2 \phi\right|^2 + |p|^2 |\phi|^2\right) dv + \tag{46}$$

$$R'_S \left(p^* \lambda + k_2 \tau\ p\right) \int\limits_V |\nabla \phi|^2 dv +$$

$$Q \int\limits_V \left[\lambda \left|curl\ \ curl\ \vec{h}\right|^2 + \frac{|p|^2 \sigma_1}{\sigma} \left|\vec{h}\right|^2\right] dv +$$

$$Q\left[\frac{p^* \lambda \sigma_1}{\sigma} + p\right] \int\limits_V \left|curl\, \vec{h}\right|^2 dv = 0$$

Equating the imaginary part of equation (46) to zero and assuming $pi \neq 0$, we get

$$\left(1+\frac{\lambda}{\sigma}\right)\int\limits_{V}|curl\ \vec{q}|^2 + \frac{2p_r}{\sigma}\int\limits_{V}|\vec{q}|^2 dv - R'_T\,(k_1-\lambda)$$
$$\int\limits_{V}|\nabla\theta|^2\,dv + R'_S\ \ (\tau\ k_2-\lambda)\int\limits_{V}|\nabla\phi|^2\,dv = 0. \tag{47}$$

Equation (42) cannot obviously be satisfied under the conditions of the theorem. Hence, we must have $p_i = 0$.

This completes the proof of the theorem.

Theorem 3 implies that for the hydromagnetic thermosolutal convection coupled with cross- diffusion, an arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular PES is valid if $k_1\sigma_1 < \sigma \leq \sigma_1 k_2\tau$. Further, the theorem also implies the validity of this result for the

i) thermosolutal convection coupled with cross-diffusion if $\tau\ > \dfrac{k_1}{k_2}$;

ii) thermosolutal convection of the Veronis' type if $\tau > 1$.

The subsequent theorem provides limitations on the complex growth rate of oscillatory motions of growing amplitude for the problem under consideration, which may obviously exit if the sufficient conditions of Theorem 3 are violated.

Theorem 4: If $(p,\vec{q},\ \theta,\ \phi,\ \vec{h})$, $p = p_r + ip_i$, $p_r \geq 0$, $p_i \neq 0$ is a non-trivial solution of equation (23)-(27) together with the boundary conditions (28) and $R'_T > 0$, $R'_S \geq 0$ and $k_2\tau < \lambda < k_1$, then for large Q (or for large R_S if Q = 0)

$$|p| < \hat{\lambda}\left[\tau\ R'_T\,(k_1-\lambda) + R'_S\,(\lambda - k_2\tau)\right].$$

where $\hat{\lambda} = \frac{l^2}{\tau\lambda_0(\sigma+\lambda)}$, λ is as in Theorem 3 and l *and* λ_0 are as in lemma 1.

Proof: It follows from equation (24) that

$$\int_{V} \left(k_1\nabla^2\theta - p\theta\right)\left(k_1\nabla^2\theta^* - p * \theta^*\right) dv = \int_{V} \left|\vec{q}\cdot\hat{k}\right|^2 dv \tag{48}$$

or

$$k_1^2\int_{V} \left|\nabla^2\theta\right|^2 dv + 2p_r k_1\int_{V} \left|\nabla\theta\right|^2 dv + |p|^2\int_{V} |\theta|^2 dv = \int_{V} \left|\vec{q}\cdot\hat{k}\right|^2 dv \tag{49}$$

(using lemma 2)

Equation (49), upon using $p_r \geq 0$, $pi \neq 0$ gives

$$\int_{V} |\theta|^2 dv < \frac{1}{|p|^2}\int_{V} \left|\vec{q}\cdot\hat{k}\right|^2 dv \leq \frac{1}{|p|^2}\int_{V} |\vec{q}|^2 dv \tag{50}$$

Again multiplying (24) by $\theta *$, integrating over the domain V, using lemma 2 and equating the real parts of the resulting equation, we have

$$k_1\int_{V} |\nabla\theta|^2 dv + p_r\int_{V} |\theta|^2 dv = Real\ part\ of\ \left(\int_{V} \theta^*\left|\vec{q}\cdot\hat{k}\right| dv\right)$$

$$\leq \left|\cdot\int_{V} \theta^*\left(\vec{q}\cdot\hat{k}\right) dv\right| \leq \int_{V} \theta^*\left|\vec{q}\cdot\hat{k}\right| dv$$

which, upon using Schwartz's inequality and the fact that $p_r \geq 0$, gives

$$\begin{aligned} k_1\int_{V} |\nabla\theta|^2 dv &\leq \cdot\left[\int_{V} |\theta|^2 dv\right]^{1/2}\left[\left(\int_{V} \left|\vec{q}\cdot\hat{k}\right|^2 dv\right)\right]^{1/2} \\ &\leq \left[\int_{V} |\theta|^2 dv\right]^{1/2}\left[\left(\int_{V} |\vec{q}|^2 dv\right)\right]^{1/2}. \end{aligned} \tag{51}$$

Combining inequalities (50) and (51), we get

$$k_1 \int\limits_{V} |\nabla\theta|^2 \, dv \; < \; \frac{1}{|p|} \int\limits_{V} |\vec{q}|^2 \, dv. \tag{52}$$

Further, the solenoidal character of the velocity field $\vec{q}$ namely $div\vec{q} = 0$, implies that

$$\int\limits_{V} |curl \; \vec{q}|^2 \, dv = \int\limits_{V} \left(\vec{q}^{*} \cdot curl \; curl \; \vec{q} \right) \, dv = -\int\limits_{V} \vec{q}^{*} \nabla^2 \vec{q} dv$$

which upon taking

$$\vec{q} = (u, v, w) \,, \; gives \int\limits_{V} |curl \; \vec{q}|^2 \, dv = \int\limits_{V} \left(|\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \right) \, dv \tag{53}$$

or

$$\int\limits_{V} |\,\vec{q}|^2 \, dv < \frac{\ell^2}{\lambda_0} \int\limits_{V} |curl \; \vec{q}|^2 \, dv \tag{54}$$

(using lemma1)

Inequality (52) and (54) implies that

$$\int\limits_{V} |\,\nabla\theta|^2 \, dv < \frac{\ell^2}{\lambda_0 k_1 \, |p|} \int\limits_{V} |curl \; \vec{q}|^2 \, dv \tag{55}$$

Similarly proceeding from equation (25), and emulating the steps in the derivation of inequality (55), we have

$$\int\limits_{V} |\,\nabla\phi|^2 \, dv < \frac{\ell^2}{\lambda_0 k_2 \tau \, |p|} \int\limits_{V} |curl \; \vec{q}|^2 \, dv \tag{56}$$

Using inequality (56) and (57) in equation (47), we get

$$\frac{(\sigma + \lambda)}{\sigma \, |p|} \left\{ |p| - \hat{\lambda} \left[\tau \; R'_T \, (k_1 - \lambda) + R'_S \, (\lambda - k_2 \tau) \right] \right\} \tag{57}$$

$$\int_v |curl\ q|^2\, dv + \frac{2p_r}{\sigma} \int_v |q|^2 dv < 0$$

Inequality (57) clearly implies that

$$|p| < \hat{\lambda}\left[\tau\ R_T'\left(k_1 - \lambda\right) + R_S'\left(\lambda - k_2\tau\right)\right].$$

This completes the proof of the theorem.

Theorem 4 implies that the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for the hydromagnetic thermosolutal convection coupled with cross-diffusion lies inside a semi- circle with centre at origin and

$$radius = \hat{\lambda}\left[\tau\ R_T'\left(k_1 - \lambda\right) + R_S'\left(\lambda - k_2\tau\right)\right],\ (\ \lambda = \frac{\sigma}{\sigma_1}\ )$$

in the right half of the complex p-plane.

3 Conclusions

The instability of thermosolutal convection coupled with cross-diffusion of an electrically conducting fluid completely confined in an arbitrary region bounded by rigid wall in the presences of a uniform magnetic field applied in an arbitrary direction is investigated in the present paper. The principal conclusions from the analysis of this study are:

1. Principle of exchange of stabilities is not valid for the hydromagnetic thermosolutal convection coupled with cross-diffusion if $\frac{\tau\ k_2}{k_1} R_T' \leq R_S'$.

2. Hydromagnetic thermosolutal convection coupled with cross- diffusion is stable if the Lewis number $\tau = 1$

3. For the hydromagnetic thermosolutal convection coupled with cross- diffusion, an arbitrary neutral or unstable mode is definitely non-oscillatory in character and in particular principle of exchange of stabilities is valid if $k_1\sigma_1 < \sigma \leq \sigma_1 k_2 \tau$.

4. The complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable for the hydromagnetic thermosolutal convection coupled with cross-diffusion lies inside a semi- circle with centre at origin and

$$radius = \hat{\lambda}\left[\tau\ R'_T\left(k_1-\lambda\right)+R'_S\left(\lambda-k_2\tau\right)\right],\ (\ \lambda=\frac{\sigma}{\sigma_1}\ )$$

in the right half of the complex p-plane.

5. The results for the thermosolutal convection problems with or without the individual consideration of Dufour and Soret effects follow as a consequence.

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Submitted on November 2010.

O hidromagnetskoj termo-rastvorljivoj konvekciji spregnutoj sa unakrsnom difuzijom potpuno zatvorenog fluida

Istražuje se nestabilnost of termo-rastvorljive konvekcije spregnute sa unakrsnom difuzijom elektroprovodnog fluida potpuno sadržanog u nekoj proizvoljnoj oblasti ograničenoj krutim zidom u prisustvu proizvoljno usmerenog uniformnog magnetskog polja. Izvedeni su neki opšti kvalitativni rezultati koji se odnose na kakrakter marginalnog stanja, stabilnost oscilatornih kretanja i ograničenja na rastuću amplitudu. Rezultati za probleme termo-rastvorljive konvekcije sa ili bez posebnog razmatranja Dufour-ovog i Soret-ovog efekta dobijaju se kao posledica.

doi:10.2298/TAM1004251M Math.Subj.Class.: 80A20, 80A32, 80M99