

ADOMIAN–HERMITE–PADÉ APPROXIMATION APPROACH TO THERMAL CRITICALITY FOR A REACTIVE THIRD GRADE FLUID FLOW THROUGH POROUS MEDIUM

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ABSTRACT. This paper investigates the effect of non-Newtonian material effect on the thermal stability of a reactive fluid flow through a channel saturated with porous medium by using Brinkman model. Approximate solution of the dimensionless nonlinear ordinary differential equation governing the fluid flow is obtained by using Adomian decomposition method together with special Hermite–Padé approximant. Effects of various non-Newtonian fluid parameters on both the velocity and temperature fields are constructed and discussed.

1. Introduction

In recent times, studies on non-Newtonian fluids are on the increase due to its numerous applications in a number of engineering and industrial applications. For instance, it occurs during polymer extrusion, fuel processing in refineries, recovery of heavy oil by in-situ combustion and in petro-chemical products like lubricating grease. With these applications in mind, the classical Navier-Stokes equations cannot accurately describe the rheological behavior of these complex fluids. To this end, the third grade fluid model has been used to model the flow of non-Newtonian fluids [1–24] undergoing strongly exothermic chemical reaction due to its ability to explain the shear thickening/thinning properties of these fluids.

Motivated by a recent study by Makinde [10] in which the flow of a reactive fluid through a channel filled with saturated porous medium was investigated, the specific objective of this work is to extend the result obtained in [10] to a wide range of non-Newtonian fluid which have many industrial and engineering applications especially in fluid flow undergoing exothermic chemical reactions. Several other applications could be found many geophysics, petrochemical industries during fractional distillation of combustible non-Newtonian fluids like crude oil at a very

2010 *Mathematics Subject Classification:* 80A20, 76S05, 76A05.

Key words and phrases: thermal criticality, third grade fluid, multiple solutions, Arrhenius kinetics.

high temperature in refineries, heat transfer during recovery of heavy oil, bitumen, polymer extrusion etc. In all the above cases spontaneous heating (thermal ignition) of the fluid could occur giving out huge amount of heat that is dangerous to both lives and properties. The essence of thermal stability analysis is to introduce a control measure so as to improve safety of working environment and the quality of products in the petro-chemical industries.

The problem under discussion is nonlinear due to the presence of exponential temperature dependent internal heat generation arising from Arrhenius kinetics within the channel. Therefore, approximate solution will be obtained using the combination of Adomian decomposition method (see [15–27]) together with Padé approximant to enhance the accuracy and the domain of convergence of the series solution. The rest of the paper consists of the following: in Section 2, the Mathematical analysis of the flow and the non-dimensionalization of the governing equation are presented. The dimensionless problem is solved in the Section 3. Results are presented and discussed in Section 4 while Section 5 concludes the paper.

2. Mathematical Analysis

Consider the steady flow of a viscous incompressible third grade fluid through infinite parallel isothermal plates of distance $2a$ apart. The channel is assumed to be saturated with porous material as shown in the geometry below. The channel walls are kept constant at temperature T_a .

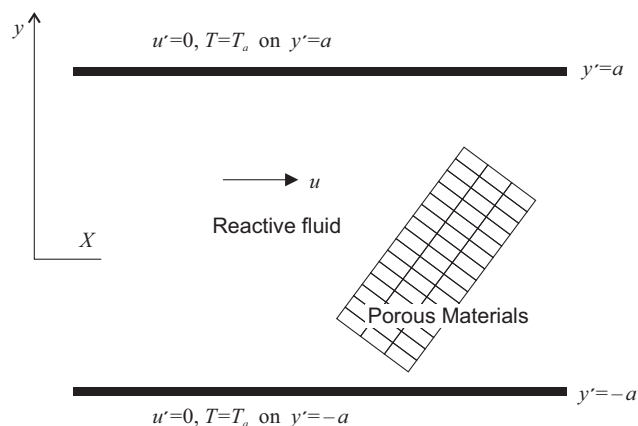


FIGURE 1. Flow Geometry

Neglecting the reactant consumption, the equations governing the hydrodynamically and thermally developed flow can be written as [1, 10].

$$(2.1) \quad 0 = -\frac{d\hat{p}}{dx} + \mu \frac{d^2 u'}{dy'^2} + 6\beta_3 \frac{d^2 u'}{dy'^2} \left(\frac{du'}{dy'} \right) - \frac{\mu}{K} \mu'$$

$$(2.2) \quad 0 = k \frac{d^2 T}{dy'^2} + \left(\frac{du'}{dy'} \right)^2 \left\{ \mu + 2\beta_3 \left(\frac{du'}{dy'} \right)^2 \right\} + \frac{\mu}{K} u'^2 + QAC_0 e^{-\frac{E}{RT}}$$

Subject to the appropriate boundary conditions

$$\begin{aligned} u' &= 0, & T &= T_a & \text{on } & y' = a, \\ u' &= 0, & T &= T_a & \text{on } & y' = -a, \end{aligned}$$

where u' is the fluid velocity, P' is the fluid pressure, β_3 is the material coefficient, k the thermal conductivity, μ is the dynamic viscosity, γ is the dimensionless third grade material parameter, T is the fluid temperature, U is the characteristic velocity, T_a the wall temperature Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the channel width, y the distance measured in the normal direction and μ the fluid dynamic viscosity coefficient. Additional term in (2.1) and (2.2) is the third grade material effect due to [2-9]. Introducing the following dimensionless parameters

$$\begin{aligned} y &= \frac{y'}{a}, & u &= \frac{u'}{U}, & \gamma &= \frac{\beta_3 U^2}{\mu a^2}, & \beta^2 &= \frac{1}{Da}, & Da &= \frac{K}{a^2}, & M &= -\frac{a^2}{\mu U} \frac{dP}{dx}, \\ \lambda &= \frac{QAEa^2 C_0 e^{-\frac{E}{RT_0}}}{kRT_a^2} & \alpha &= \frac{\mu U^2 e^{\frac{E}{RT_0}}}{QAC_0 a^2}, & \varepsilon &= \frac{RT_a}{E}, & \theta &= \frac{E(T - T_a)}{RT_a^2}, \end{aligned}$$

we obtain the dimensionless equations with appropriate boundary conditions as follows

$$(2.3) \quad \frac{d^2 u}{dy^2} + 6\gamma \frac{d^2 u}{dy^2} \left(\frac{du}{dy}\right)^2 - \beta^2 u = -M; \quad u(-1) = 0 = u(1),$$

$$(2.4) \quad \frac{d^2 \theta}{dy^2} + \lambda \left[e^{\frac{\theta}{1+\varepsilon\theta}} + \alpha \left(\frac{du}{dy}\right)^2 \left(1 + 2\gamma \left(\frac{du}{dy}\right)^2\right) + \alpha\beta^2 u^2 \right] = 0; \quad \theta(-1) = 0 = \theta(1)$$

Where α viscous heating parameter, θ is the dimensionless temperature, u is the dimensionless velocity, M is the dimensionless pressure gradient, β^2 is the porous permeability parameter, Da is the Darcy parameter, K porous permeability, a is the half width of the channel, λ is the Frank-Kamenetskii parameter and $\varepsilon = 0$ at extremely high temperature.

3. Method of Solution

To obtain the solution of (2.3)-(2.4), we first convert it to the integral form

$$\begin{aligned} u(y) &= u(0) + \int_0^y \int_0^y \left\{ \beta^2 u - M - 6\gamma \frac{d^2 u}{dY^2} \left(\frac{du}{dY}\right)^2 \right\} dY dY \\ \theta &= \theta(0) + \int_0^y \int_0^y \lambda \left\{ e^{\frac{\theta}{1+\varepsilon\theta}} + \alpha \left(\frac{du}{dY}\right)^2 \left(1 + 2\gamma \left(\frac{du}{dY}\right)^2\right) + \alpha\beta^2 u^2 \right\} dY dY \end{aligned}$$

If we assume a series solution in the form

$$u(y) = \sum_{n=0}^{\infty} u_n(y), \quad \theta(y) = \sum_{n=0}^{\infty} \theta_n(y)$$

Then by successive approximation, each term of the series can be obtained by using the recurrence relation.

$$u_0(y) = a_0 - \int_0^y \int_0^y (M) dY dY$$

$$u_{n+1}(y) = \int_0^y \int_0^y (\beta^2 u_n - 6\gamma B_n) dY dY \quad n \geq 0$$

and

$$\theta_0 = b_0 - \int_0^y \int_0^y \lambda \left\{ \alpha \left(\frac{du}{dY} \right)^2 \left(1 + 2\gamma \left(\frac{du}{dy} \right)^2 \right) + \alpha \beta^2 u^2 \right\} dY dY$$

$$\theta_{n+1}(y) = - \int_0^y \int_0^y \lambda e^{\frac{\theta}{1+\varepsilon\theta}} dY dY$$

Here $a_0 = \frac{du(0)}{dY}$, $b_0 = \frac{d\theta(0)}{dY}$ are to be determined using the boundary condition $u(1) = 0$, $\theta(1) = 0$ respectively. The nonlinear terms represented by

$$C = e^{\frac{\theta}{1+\varepsilon\theta}}, \quad B = \frac{d^2 u}{dY^2} \left(\frac{du}{dY} \right)^2$$

are decomposed into Adomian polynomial as follows

$$C_0 = e^{\frac{\theta}{1+\varepsilon\theta_0}}$$

$$C_1 = \frac{\theta}{1 + \varepsilon\theta} e^{\frac{\theta}{1+\varepsilon\theta_0}}$$

$$C_2 = \frac{\{(1 - 2\varepsilon - 2\varepsilon^2\theta_0)\theta_1^2 + 2(1 + \varepsilon(\theta_0)^2\theta_2)\}}{2(1 + \varepsilon\theta_1)^4} e^{\frac{\theta}{1+\varepsilon\theta_0}}$$

...

together with

$$(3.1) \quad B_0 = \frac{d^2 u_0}{dY^2} \left(\frac{du_0}{dY} \right)^2$$

$$B_1 = 2 \frac{d^2 u_0}{dY^2} \left(\frac{du_0}{dY} \right) \left(\frac{du_1}{dY} \right) + \frac{d^2 u_1}{dY^2} \left(\frac{du_0}{dY} \right)^2$$

...

The approximate solution are obtained by the partial sum

$$(3.2) \quad \theta = \sum_{n=0}^m \theta_n, \quad u = \sum_{n=0}^m u_n.$$

Invoking the convergence analysis in [14], the iteration (3.1) and (3.2) is repeated until convergence is reached as presented in Table 1.

4. Bifurcation Study

Taking advantage of the quadratic Shafer approximant, it is easy to construct the upper and the lower branches of the solution. Assuming the condition $\lambda < \lambda_c$ holds, by obtaining the first $d - 1$ derivative of $\theta(\lambda)$. Let $d \in \mathbb{N}$ and let the $d + 1$ series

$$\theta_0(\lambda), \theta_1(\lambda), \theta_2(\lambda), \theta_{d-1}(\lambda), \dots, \theta_d(\lambda)$$

be given. Then, given the $(d + 1)$ -tuple polynomial

$$A_{0N}^{(0)}, A_{1N}^{(d)}, \dots, A_{2N}^{(d)}$$

Then the Hermite-Padé polynomial is defined by

$$F_d(\lambda, \theta_{N-1}) = A_N^{(0)} + A_{iN}^{(d)}(\lambda)\theta_i(\lambda) \quad \text{as } \lambda \rightarrow \infty,$$

where $d \geq 1$, $i = 1, 2, 3$ while the condition

$$(4.1) \quad A_{0N}^{(0)}(\lambda) = 1, \quad A_{iN}(\lambda) = \sum_{j=1}^{d+i} b_{ij} \lambda^{j-1}$$

ensure that the order of the series $A_{iN}(\lambda)$ increases with increase in i, d . Then the algebraic approximant defined by

$$d \geq 2, \quad \theta_0(\lambda) = 1, \quad \theta_1(\lambda) = \theta(\lambda), \quad \theta_2(\lambda) = \theta^2$$

gives the cubic Padé approximant which allows us to examine the solution branches. Assuming the condition $\lambda > \lambda_c$ holds, then a sufficient condition for the non-existence of solution will be to determine the dominant behavior of the partial sum (4.1) such that the accuracy of the λ_c will determine the accuracy of the solution. It is well known that the dominant behavior of a solution of a linear ordinary differential equation can be written as

$$u(\lambda) = \begin{cases} f(\lambda_c - \lambda)^\alpha & \text{for } \alpha \neq 0, 1, 2, \dots \\ f(\lambda_c - \lambda)^\alpha \ln |\lambda_c - \lambda| & \text{for } \alpha = 0, 1, 2, \dots \end{cases}$$

As $\lambda \rightarrow \lambda_c$, where λ_c is the critical point with critical exponent α and f is a constant. It then follows that the differential approximant can be obtained by setting

$$d \geq 2, \quad \theta_0(\lambda) = 1, \quad \theta_1(\lambda) = \theta(\lambda), \quad \theta_2(\lambda) = D\theta, \dots, \theta_d(\lambda) = D^{d-1}\theta,$$

where the differential operator is defined by $D := \frac{d}{d\lambda}$. Assuming an algebraic type, the exponent may be computed by using

$$\alpha_N = 1 - \frac{A_{2N}(\lambda_{CN})}{DA_{3N}(\lambda_{CN})}$$

The physical quantities of interest in this problem are the skin-friction parameter (C_f) and the Nusselt number (Nu) which are defined by

$$C_f = \frac{al_w}{\mu U} = -\frac{du}{dy}(1),$$

$$\text{Nu} = \frac{aEQ_w}{KRT_a^2} = -\frac{d\theta}{dy}(1),$$

5. Results and discussions

Table 1 showing convergence for the velocity profile when $m = 6, \gamma = 0$. Table 2 shows that the thermal criticality values λ_C decrease with increasing pressure gradient (M) and Brinkman number (α) but increases with increasing non-Newtonian parameter (γ) and activation energy parameter (ε). This implies that thermal stability is enhanced with increasing γ and ε , while an increase in M and α may lead to early occurrence of thermal runaway in the flow field. Figure 2 shows the effect of the non-Newtonian parameter on the velocity profile, as observed from the plot an increase in the non-Newtonian material parameter has a decreasing effect on the velocity profile due to fluid thickening. Similarly, rise in the porous permeability parameter implies a decrease in the porous permeability of the channel.

This eventually leads to decrease in the flow velocity as observed in Figure 3. Moreover, Figure 4 depicts the variation of activation energy parameter with the temperature distribution within the channel. From the graph, it is observed that fluid temperature decreases with rise in the fluid activation energy parameter.

TABLE 1. Convergence for the velocity profile when $m = 6, \gamma=0$

y	β	u_{Exact}	u_{Adomian}	Absolute Error
0	1	0	0	0
0.1	1	0.0412846	0.0412846	2.60625×10^{-14}
0.2	1	0.072941	0.072941	5.25968×10^{-14}
0.3	1	0.0953855	0.0953855	7.95197×10^{-14}
0.4	1	0.108743	0.108743	1.07123×10^{-13}
0.5	1	0.113181	0.113181	1.35905×10^{-13}
0.6	1	0.108743	0.108743	1.65895×10^{-13}
0.7	1	0.0953855	0.0953855	1.96593×10^{-13}
0.8	1	0.0729741	0.0729741	2.20796×10^{-13}
0.9	1	0.0412846	0.0412846	2.03795×10^{-13}
1.0	1	0	0	0

TABLE 2. Thermal criticality values for $\beta=1$.

M	α	γ	ε	λ_c
0.1	0.1	0.1	0	0.8780610
1.0	0.1	0.1	0	0.8725029
2.0	0.1	0.1	0	0.8579873
0.1	0.5	0.1	0	0.8778254
0.1	1.0	0.1	0	0.8775309
0.1	0.1	0.5	0	0.8780611
0.1	0.1	0.1	0	0.8780613
0.1	0.1	0.1	0.1	0.9877247
0.1	0.1	0.1	0.2	1.1613088

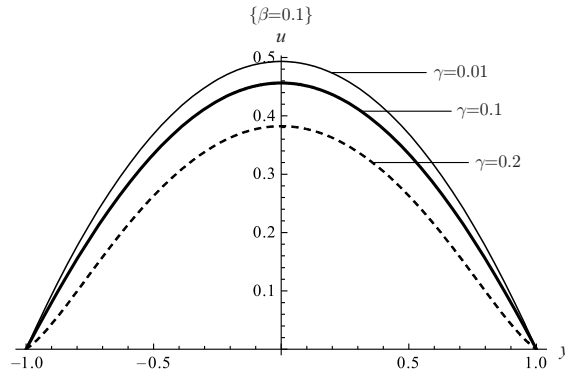


FIGURE 2. Effect of non-Newtonian material effect on velocity profile

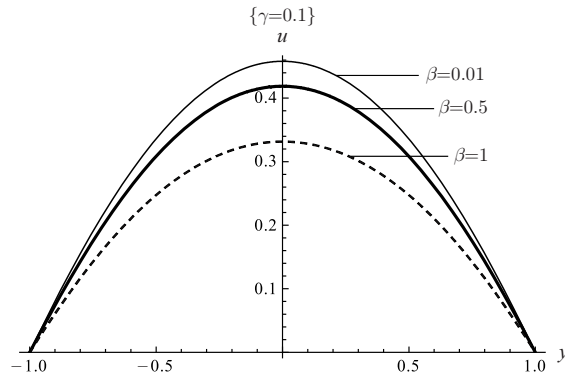


FIGURE 3. Effect of porous permeability on the velocity profile

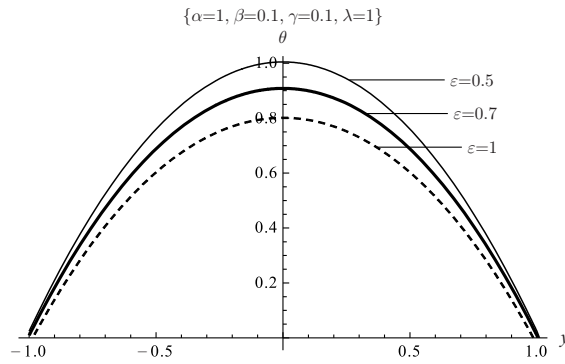


FIGURE 4. Effect of activation energy parameter on temperature profile

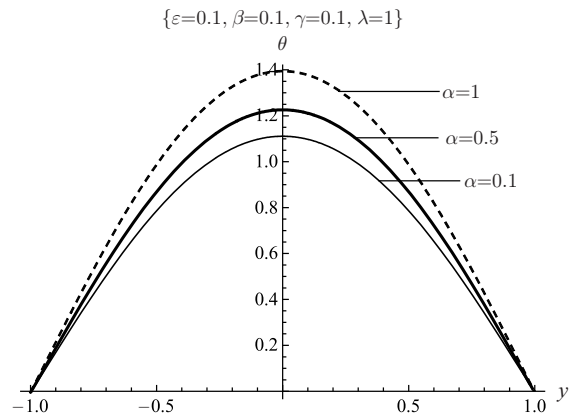


FIGURE 5. Effect of viscous heating parameter on temperature profile

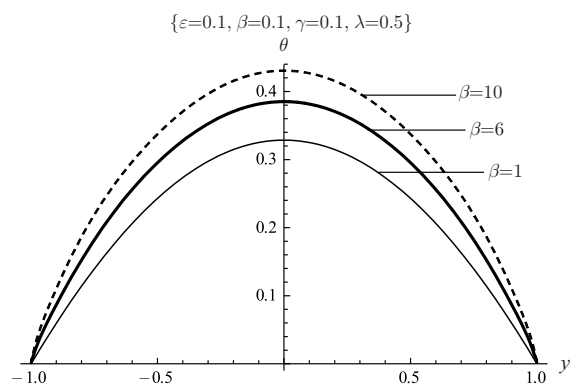


FIGURE 6. Effect of porous permeability parameter on temperature profile

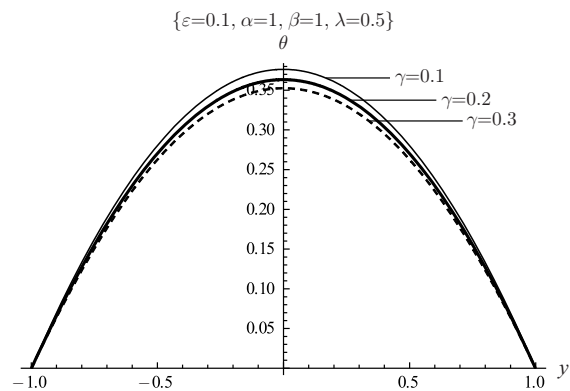


FIGURE 7. Effect of non-Newtonian material effect on temperature profile

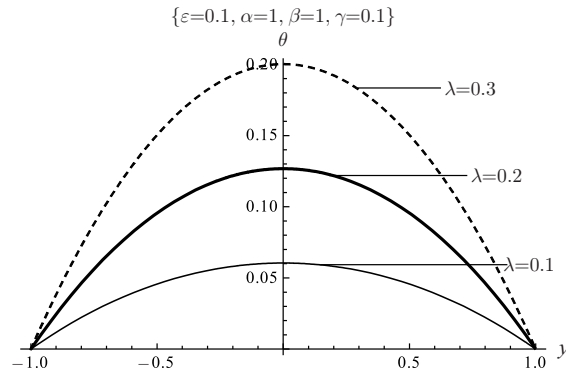


FIGURE 8. Effect of Frank–Kamenetskii parameter on temperature profile

This is due to the fact that activation energy of the fluid decreases with rise in fluid viscosity. In Figure 5, fluid temperature is observed to increase with increase in the viscous heating parameter as a result of rise in the kinetic energy of the fluid. This is physically true since viscous dissipation is an additional heat source within the channel. A plot showing the effect of porous permeability on the fluid temperature is shown in Figure 6. From the result, it is observed that rise in the porous permeability enhances rise in the fluid temperature. Physically, this is true due to decrease in the porous permeability of the channel. This means a reduced flow and more heat will be generated due to accumulation. While Figure 7 shows that the non-Newtonian material parameter is a decreasing function of temperature. In Figure 8, it is observed that for small parameter values, an increase in the Frank–Kamenetskii parameter leads to rise in the fluid temperature due to internal heat generation as a result of the exothermic chemical reactions. It is important to remark here that spontaneous heating of the fluid (thermal ignition) could occur if the rate at which heat is generated is greater than heat dissipation rate i.e., as the Frank–Kamenetskii parameter becomes large.

6. Conclusion

In this paper, the reactive third grade fluid flow through a porous channel is studied by using rapidly convergent Adomian decomposition method together with special Hermite–Padé approximant. Convergence analysis is conducted and presented in Table 1 so as to justify the accuracy of the approximation made. The main contribution to knowledge in this paper are as follows:

- (a) an increase in the non-Newtonian material parameter is decreases both the fluid flow velocity and temperature distribution within the flow channel.
- (b) an increase in the non-Newtonian material effect has destabilizing effect on the fluid flow.

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**ADOMIAN–HERMITE–PADE АПРОКСИМАЦИЈА ЗА
ТЕРМИЧКУ КРИТИЧНОСТ ТОКА РЕАКТИВНОГ
ФЛУИДА КРОЗ ПОРОЗНИ МАТЕРИЈАЛ**

РЕЗИМЕ. Испитује се утицај не-Њутновских материјалних ефеката на термичку стабилност тока реактивног флуида кроз канал испуњен порозним материјалом користећи Бринкманов модел. Апроксимативно решење бездимензионе нелинеарне диференцијалне једначине којом се описује ток флуида добија се применом методе Adomain-декомпозије заједно са специјалном Hermite–Pade апроксимацијом. Изведени су и дискутовани ефекти различитих не-Њутновских параметара флуида на брзину и температурно поље.

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(Received 28.10.2015)
(Revised 21.05.2016)
(Available online 29.06.2016)

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