ADMISSIBILITY OF A SOLUTION TO GENERALIZED CHAPLYGIN GAS

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Abstract. It is known that there is a solution to the Riemann problem for generalized Chaplygin gas model and that it contains the Dirac delta function in some cases. In some cases, usual admissible criteria can not extract a unique weak solution as it was shown in [4]. The aim of this paper is to use a solution to perturbed generalized Chaplygin model by a small constant $\varepsilon > 0$ and obtain a its unique limit. A weak solution to the unperturbed system that equals that limit is called admissible. The perturbation is made by using the modified model of Chaplygin gas defined in [5].

1. Introduction

The original Chaplygin system
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \]
\[ \frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 - \frac{1}{\rho}) = 0 \]

was introduced as a model for a fluid passing by an obstacle (see [2]). The model of generalized Chaplygin gas appears in a number of cosmology theories as a compressible fluid with a pressure inversely proportional to a gas energy density, $p = -\frac{C}{\rho^\alpha}$, $C > 0$, $0 < \alpha < 1$, see [1]. It is used as a model for the dark energy in the Universe. (We will use $C = 1$ in the rest of the paper for simplicity.) The system consists of the mass and momentum conservation laws
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \]
\[ \frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 - \frac{1}{\rho^\alpha}) = 0. \]

Let us briefly give the properties of the system. It is a strictly hyperbolic system with the eigenvalues $\lambda_1 = u - \sqrt{\alpha\rho^{-\frac{1+\alpha}{2}}} \cdot T$, $\lambda_2 = u + \sqrt{\alpha\rho^{-\frac{1+\alpha}{2}}} \cdot T$ and appropriate eigenvectors $r_1 = (-1, -u + \sqrt{\alpha\rho^{-\frac{1+\alpha}{2}}}, 1, 1) \cdot T$ and $r_2 = (1, u + \sqrt{\alpha\rho^{-\frac{1+\alpha}{2}}}, 1, 1) \cdot T$. Both fields are genuinely nonlinear.

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Using the standard procedures one can find rarefaction curves:

\[ R_1 : u = u_0 + \frac{2\sqrt{\alpha}}{1 + \alpha} \left( \rho^{\frac{1}{1+\alpha}} - \rho_0^{\frac{1}{1+\alpha}} \right), \quad \rho < \rho_0 \]

\[ R_2 : u = u_0 - \frac{2\sqrt{\alpha}}{1 + \alpha} \left( \rho^{\frac{1}{1+\alpha}} - \rho_0^{\frac{1}{1+\alpha}} \right), \quad \rho > \rho_0, \]

and shock ones:

\[ S_1 : u = u_0 - \sqrt{\frac{\rho - \rho_0}{\rho_0} \left( \frac{1}{\rho_0^{\alpha}} - \frac{1}{\rho^{\alpha}} \right)}, \quad \rho > \rho_0. \]

\[ S_2 : u = u_0 - \sqrt{\frac{\rho - \rho_0}{\rho_0} \left( \frac{1}{\rho_0^{\alpha}} - \frac{1}{\rho^{\alpha}} \right)}, \quad \rho < \rho_0. \]

Shock speeds for points at both curves \( S_1 \) and \( S_2 \) are given by

\[ c = u_0 \pm \sqrt{\frac{\rho_0^{1-\alpha} \rho^{1+\alpha} - \rho_0^{1+\alpha}}{\rho_0^{1+\alpha} \rho - \rho_0}}. \]

where + sign is for \( S_2 \) and − for \( S_1 \). A solution to the Riemann problem

\[ (\rho, u)|_{t=0} = \begin{cases} 
(\rho_0, u_0), & x < 0 \\
(\rho_1, u_1), & x > 0 
\end{cases} \]

for (1.1) is given as a combination of the elementary waves for the points \((\rho_1, q_1)\) above and on the curve

\[ (1.3) \quad \Gamma_{ss} = \Gamma_{ss}(\rho_0, q_0) : q = \left( \frac{q_0}{\rho_0} - \rho^{\frac{1}{1+\alpha}} \right) \rho. \]

Below that line there are no classical solutions. One can use shadow waves [3] in order to solve the problem there. These waves are approximate solutions to balance law systems. In [4] the following lemma is proved

**Lemma 1.1.** There exists a simple shadow wave written in the form

\[ (1.4) \quad (\rho, u)(x, t) = \begin{cases} 
(\rho_0, u_0), & x < (c - \varepsilon)t \\
(\rho_0, u_0), & (c - \varepsilon)t < x < ct \\
(\rho_1, u_1), & ct < x < (c + \varepsilon)t \\
(\rho_1, u_1), & x > (c + \varepsilon)t 
\end{cases} \]

that solves (1.1) with the initial data (1.2) if and only if

\[ (1.5) \quad (u_0 \rho_0 \rho_1 - u_1 \rho_0 \rho_1)^2 > (\rho_0 - \rho_1) \left( \frac{1}{\rho_1^{1+\alpha}} - \frac{1}{\rho_0^{1+\alpha}} \right) \rho_0 \rho_1. \]

The speed \( c \) is given by

\[ c = \frac{[\rho u] + \sqrt{[\rho u]^2 - [\rho] \left( \frac{(\rho u)^2 - \rho^{1-\alpha}}{\rho} \right)}}{[\rho]}, \]
Figure 1. Classical waves

where \([x]\) denotes a jump \(x_1 - x_0\). The strength of the shadow wave equals

\[
\sigma = \sqrt{(u_0 - u_1)^2 \rho_0 \rho_1 - (\rho_0 - \rho_1) \left( \frac{1}{\rho_1^2} - \frac{1}{\rho_0^2} \right)}.
\]

The solution (1.4) converges to

\[
(\bar{\rho}, \bar{u})(x, t) = \begin{cases} 
(\rho_0, u_0), & x < ct \\
(\rho_1, u_1), & x > ct, \end{cases} + \sigma t,
\]

in the distributional sense as \(\varepsilon \to 0\).

“Solve” here means that a distributional limit of the left–hand sides in (1.1) with \((\rho, u)\) substituted by a net (1.4) equals zero as \(\varepsilon \to 0\).

In order to get a unique solution to the given Riemann problem, one has to exclude all points \((\rho_1, u_1)\) satisfying (1.5) above the line \(\Gamma_{ss}\) by an admissibility criterion. The usual one, overcompressibility, \(\lambda_2(\rho_0, u_0) \geq \lambda_1(\rho_0, u_0) \geq c \geq \lambda_2(\rho_1, u_1) \geq \lambda_1(\rho_1, u_1)\), is not enough. In [4], one can find a better condition made by using Lax entropy pairs, but there are still points above \(\Gamma_{ss}\) for which one cannot know whether they are admissible or not.

Our aim is to perturb the second equation in (1.1) by adding a small term depending on \(\varepsilon\) such that we get a modified Chaplygin model described in [5]. Then, we expect that \(\varepsilon \to 0\) will recover the above shadow wave solution only below the curve \(\Gamma_{ss}\). That could be used as a new admissibility criterion. One can see a similar procedure for the original Chaplygin system in [6].
2. Modified Chaplygin gas model

We will use the following perturbation of the system (1.1) based on the model from [5] by letting the positive parameter in the flux function vanish

\[ \partial_t \rho + \partial_x (\rho u) = 0 \]

\[ \partial_t (\rho u) + \partial_x \left( \rho u^2 + \varepsilon \rho - \frac{1}{\rho^\alpha} \right) = 0. \]

Let us assume initial data (1.2) for the system. Such Riemann problem has a unique solution in the physical domain \( \rho > 0, \ u \in \mathbb{R} \) given as a combination of elementary waves. Let us briefly describe those solutions.

We have a strictly hyperbolic system with the eigenvalues

\[ \lambda_1 = u - \sqrt{\varepsilon + \alpha \rho^{-(1+\alpha)}}, \]
\[ \lambda_2 = u + \sqrt{\varepsilon + \alpha \rho^{-(1+\alpha)}} \]

and appropriate eigenvectors \( r_1 = (-1, -u + \sqrt{\varepsilon + \alpha \rho^{-(1+\alpha)}})^T \) and \( r_2 = (1, u + \sqrt{\varepsilon + \alpha \rho^{-(1+\alpha)}})^T \). Both fields are genuinely nonlinear for \( \alpha \in (0, 1) \) and \( \varepsilon \) small enough.

Perturbed rarefaction curves are given by

\[ R_{1, \varepsilon} : u = u_0 - \int_{\rho_0}^{\rho} s^{-1} \sqrt{\varepsilon + \alpha s^{-(1+\alpha)}} ds, \quad \rho < \rho_0 \]
\[ R_{2, \varepsilon} : u = u_0 + \int_{\rho_0}^{\rho} s^{-1} \sqrt{\varepsilon + \alpha s^{-(1+\alpha)}} ds, \quad \rho > \rho_0. \]

Perturbed shock curves are

\[ S_{1, \varepsilon} : u = u_0 - \frac{1}{\sqrt{\rho_0 \rho}} \left( (\rho_0^{-\alpha} - \rho^{-\alpha}) + \varepsilon (\rho - \rho_0) \right), \quad \rho > \rho_0, \]
\[ S_{2, \varepsilon} : u = u_0 - \frac{1}{\sqrt{\rho_0 \rho}} \left( (\rho_0^{-\alpha} - \rho^{-\alpha}) + \varepsilon (\rho - \rho_0) \right), \quad \rho < \rho_0. \]

A shock speed for a point \((\rho, u)\) at the curve \( S_1 \) or \( S_2 \) is given by

\[ c_{1, \varepsilon} = u_0 - \sqrt{\frac{\rho_0^{-\alpha} - \rho^{-\alpha}}{\rho_0^{-\alpha} - \rho^{-\alpha}} + \varepsilon}, \quad \text{or} \quad c_{2, \varepsilon} = u_0 + \sqrt{\frac{\rho_0^{-\alpha} - \rho^{-\alpha}}{\rho_0^{-\alpha} - \rho^{-\alpha}} + \varepsilon}. \]

A solution to the Riemann problem

\[ (\rho, u)|_{t=0} = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0 \end{cases} \]

for (2.1) is given as a combination of the elementary waves for all points \((\rho, u) \in \mathbb{R}_+ \times \mathbb{R}\) contrary to the case of generalized Chaplygin gas (1.1). That is the main difference between (1.1) and (2.1). One can find an illustration in Figure 2: grey lines represent non-perturbed rarefaction and shock curves, the black line is \( \Gamma_{ss} \).

Note the important fact: \( S_{1, \varepsilon} \) crosses the line \( \Gamma_{ss} \).

One could see that all perturbed \( R \) and \( S \)–curves tend to unperturbed ones, but with one significant difference. The perturbed \( S_2 \)–curve lies sufficiently below the critical curve \( \Gamma_{ss} \) so one could expect that it could be possible to obtain a \( S_1 + S_2 \) solution to (2.1) below the critical curve. One can see an illustration in Figure 3. Unlike the original system we have the following lemma.
**Theorem 2.1.** The Riemann problem (2.1), (1.2) has a unique entropic solution consisting of a combination of shocks and rarefaction waves.

As \( \varepsilon \to 0 \) the solution tends to the one to (1.1). Additionally, it converges to the shadow wave solution (1.4) if and only if \((\rho_1, u_1)\) lies below the curve \( \Gamma_{ss} \).

**Proof.** Proof for all areas but the one between \( S_1, \varepsilon \) and \( S_2, \varepsilon \) curve is almost the same as for the original system (1.1). We will present a proof for that area here.

First, we will prove that any point \((\rho_1, u_1)\) between these lines can be connected to \((\rho_0, u_0)\) by a combination of two shocks. Let us denote a middle state by \((\rho_\varepsilon, u_\varepsilon)\).

It is a solution to the following system of equations

\[
\begin{align*}
    u_\varepsilon &= u_0 - \frac{1}{\sqrt{\rho_1 \rho_\varepsilon}} \sqrt{(\rho_\varepsilon - \rho_0)((\rho_0^{\alpha} - \rho_\varepsilon^{\alpha}) + \varepsilon(\rho_\varepsilon - \rho_0))}, \quad \rho_\varepsilon > \rho_0, \\
    u_1 &= u_\varepsilon + \frac{1}{\sqrt{\rho_1 \rho_\varepsilon}} \sqrt{(\rho_1 - \rho_\varepsilon)((\rho_\varepsilon^{\alpha} - \rho_1^{\alpha}) + \varepsilon(\rho_1 - \rho_\varepsilon))}, \quad \rho_1 < \rho_\varepsilon.
\end{align*}
\]

Thus, the value \( \rho_\varepsilon \) is a solution to the equation \( f_1(\rho) = f_2(\rho) \), where

\[
f_1(\rho) := u_0 - \sqrt{\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) \left(\frac{1}{\rho_0^{\alpha}} - \frac{1}{\rho^{\alpha}}\right) + \varepsilon \left(\frac{\rho}{\rho_0} + \frac{\rho_0}{\rho} - 2\right)}
\]
and

\[ f_2(\rho) := u_1 + \sqrt{\left( \frac{1}{\rho_1} - \frac{1}{\rho} \right) \left( \frac{1}{\rho_1^2} - \frac{1}{\rho^2} \right) + \varepsilon \left( \frac{\rho}{\rho_1} + \frac{\rho_1}{\rho} - 2 \right)}. \]

One can see that \( f'_1(\rho) < 0, \rho > \rho_0, f_1(\infty) = -\infty, f'_2(\rho) > 0, \rho > \rho_1, \) and \( f_1(\infty) = \infty \) so there exists a unique solution \( \rho_\varepsilon \) to the equation \( u_\varepsilon = f_1(\rho_\varepsilon) = f_2(\rho_\varepsilon). \) Immediately, one sees that \( \rho_\varepsilon \) is increasing and goes to infinity like \( \text{const} / \varepsilon \) as \( \varepsilon \to 0. \) The value of \( u_\varepsilon \) is then uniquely defined, too.

We will now prove that a distributional limit of a shock combination \( S_{1,\varepsilon} + S_{2,\varepsilon} \) is the same as a limit of the shadow wave solution to the unperturbed system (1.1).

Denote by \( c_{1,\varepsilon} \) a speed of a shock wave connecting \((\rho_0, u_0)\) with \((\rho_\varepsilon, u_\varepsilon)\) and by \( c_{2,\varepsilon} \) a speed of a shock wave connecting \((\rho_\varepsilon, u_\varepsilon)\) with \((\rho_1, u_1)\). Using the relations

\[ c_{1,\varepsilon} = \frac{\rho_\varepsilon u_\varepsilon - \rho_0 u_0}{\rho_\varepsilon - \rho_0}, \quad c_{2,\varepsilon} = \frac{\rho_\varepsilon u_\varepsilon - \rho_1 u_1}{\rho_\varepsilon - \rho_1}, \]

we have

\[ \bar{\sigma} := \lim_{\varepsilon \to 0} (c_{2,\varepsilon} - c_{1,\varepsilon}) \rho_\varepsilon = \frac{\rho_\varepsilon}{(\rho_\varepsilon - \rho_0)(\rho_\varepsilon - \rho_1)} \left( (\rho_0 u_0 - \rho_1 u_1) \rho_\varepsilon + (\rho_1 - \rho_0) \rho_\varepsilon u_\varepsilon \right) \]

\[ = (\rho_1 - \rho_0) \lim_{\varepsilon \to 0} u_\varepsilon - (\rho_1 u_1 - \rho_0 u_0) \]

Letting \( \varepsilon \to 0 \), we get \( \lim_{\varepsilon \to 0} u_\varepsilon = \lim_{y \to \infty} f_1(\rho_\varepsilon) = u_0 - \rho_0^{-1/2}(\rho_0^{-\alpha} + y)^{1/2} \) and \( \lim_{\varepsilon \to 0} u_\varepsilon = \lim_{y \to \infty} f_2(\rho_\varepsilon) = u_1 - \rho_1^{-1/2}(\rho_1^{-\alpha} + y)^{1/2} \), where \( y = \lim_{\varepsilon \to 0} \varepsilon \rho_\varepsilon \). One can find \( y \) from these relations, \( y = \xi^2 - \rho_0^{-\alpha} \), where

\[ \xi = \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \left( \frac{u_0 - u_1}{\sqrt{\rho_0}} - \frac{1}{\sqrt{\rho_1}} \sqrt{(u_0 - u_1)^2 - \left( \frac{1}{\rho_0} - \frac{1}{\rho_1} \right) \left( \frac{1}{\rho^2_0} - \frac{1}{\rho^2_1} \right)} \right). \]
Finally, that gives

$$
\tilde{\sigma} = \sqrt{\rho_0 \rho_1} \left( (u_0 - u_1)^2 + \left( \frac{1}{\rho_0} - \frac{1}{\rho_1} \right) \left( \frac{1}{\rho_0} - \frac{1}{\rho_1} \right) \right)
$$

that equals $\sigma$ from (1.6). Also $\lim_{\epsilon \to 0} c_{i, \epsilon} = \lim_{\epsilon \to 0} u_{\epsilon} = c$, $i = 1, 2$, with $c$ from (1.6). That means that the distributional limit of the solution to (2.1) for $(\rho_1, u_1)$ below $\Gamma_{ss}$ equals the distributional limit of (1.4). Above that curve the limit is $S_1 + S_2$. □

Thus, this theorem could be used as an admissibility condition for eliminating unwanted shadow waves above the curve $\Gamma_{ss}$.

An approximate solution to (1.1) is admissible if and only if a classical solution to (2.1) converges to the same distribution.

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References

ПРИХВАТЉИВОСТ РЕШЕЊА УОПШТЕНОГ ЧАПЛИГИНОВОГ ГАСА

Резиме. Познато је да постоје решења Римановог проблема за уопштени Чаплигинов гас која садрже делта функцију. У неким ситуацијама не могу да се примене уобичајене методе бирања јединственог слабог решења, како је то показано у [4]. Циљ овог рада је да се искористи ограничено решење пертурбованог уопштеног Чаплигиновог модела малом константом $\varepsilon > 0$ и нађе јединствени лимит таквог решења када $\varepsilon \to 0$. Слабо решење непертурбованог система којем тежи то решење пертурбованог система ће бити прихватљиво.

Пертурбација је урађена коришћењем модела модификованог Чаплигиновог гаса дефинисаног у [5].

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