The stability of discrete systems with time-varying delay is considered. Some sufficient delay-dependent stability conditions are derived using an appropriate model transformation of the original system. The criteria are presented in the form of LMI, which are dependent on the minimum and maximum delay bounds. It is shown that the stability criteria are approximately the same conservative as the existing ones, but have much simpler mathematical form. The numerical example is presented to illustrate the applicability of the developed results.

Keywords: time-delay systems; interval time-varying delay; asymptotic stability; delay-dependent stability; Lyapunov method.
numerical example is included to show that our results are approximately the same conservative as the existing ones [23], but have much simpler mathematical presentation. Our stability condition is expressed by means of one LMI, while the stability condition in [23] is presented by two LMI.

**Notation.** \( \mathbb{R}^n \) and \( \mathbb{Z}^+ \) denote the \( n \)-dimensional Euclidean space and positive integers. Notation \( P \succ 0 \) (\( P \succeq 0 \)) means that matrix \( P \) is real, symmetric and positive definite (semi-definite). For real symmetric matrices \( P \) and \( Q \), the notation \( P \succ Q \) (\( P \succeq Q \)) means that matrix \( P - Q \) is positive definite (positive semi-definite). \( I \) is an identity matrix with an appropriate dimension. Superscript "\( T \)" represents the transpose. In symmetric block matrices or complex matrix expressions, we use an asterisk (*) to represent a term which is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

**PROBLEM FORMULATION AND SOME PRELIMINARIES**

Consider a linear discrete-time varying delay in the state:

\[
\begin{align*}
    x(k+1) &= Ax(k) + Bx(k-h(k)) \\
    x(\theta) &= \psi(\theta), \theta \in [-h_m, -h_m + 1, \ldots, 0] \triangleq \Delta
\end{align*}
\]

with an associated function of initial state:

\[
    x(\theta) = \psi(\theta), \theta \in [-h_m, -h_m + 1, \ldots, 0] \triangleq \Delta
\]

where \( x(k) \in \mathbb{R}^n \) is the state at instant \( k \), matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \) are constant matrices. \( h(k) \) is the positive integer representing the time delay of the system that we assume to be time dependent and satisfies the following:

\[
    h_m \leq h(k) \leq h_u
\]

where \( h_m \) and \( h_u \) are constant positive integer representing the minimum and maximum delays, respectively.

The assumption on the time delay \( h(k) \) in Inequality (3) characterises the real situation in many practical applications. A typical example containing time delays that can be characterised by Inequality (3) can be found in networked control system, where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying, and can be assumed to have minimum and maximum delay bounds without loss of generality.

Let \( D(\Delta, \mathbb{R}^n) \) is space of functions mapping the discrete interval \( \Delta \) into \( \mathbb{R}^n \). Then, for \( \psi(\theta) \in D \):

\[
    \|\psi\|_D = \sup_{\theta \in \Delta} \|\psi(\theta)\|
\]

is the norm of an element \( \psi \in D \) in space \( D \).

Let \( \mathbb{R}^r = \{ \psi \in D : \|\psi\|_D < r, r \in \mathbb{R} \} \subset D \).

**Definition 1** [16]. The equilibrium state \( x = 0 \) of system given by Eq. (1) is asymptotically stable if any initial \( \psi(\theta) \) which satisfies:

\[
    \psi(\theta) \in D^-
\]

holds:

\[
    \lim_{k \to \infty} x(k, \psi) \to 0
\]

**Lemma 1** [16]. If there exist positive numbers \( \alpha \) and \( \beta \) and continuous functional \( V : D \to \mathbb{R} \) such that

\[
    0 < V(x_k) \leq \alpha \|x_k\|_D^2, \quad \forall x_k \neq 0, \quad V(0) = 0
\]

\[
    \Delta V(x_k) \equiv V(x_{k+1}) - V(x_k) \leq -\beta \|x(k)\|_D^2
\]

\[
    \forall x_k \in D
\]

satisfying Eq. (1), then the solution \( x = 0 \) of Eqs. (1) and (2) is asymptotically stable.

**Definition 2** [16]. Discrete system with time delay given by Eq. (1) is asymptotically stable if and only if its solution \( x = 0 \) is asymptotically stable.

**MAIN RESULTS**

In this section, we aim to establish an asymptotic stability criterion for the system given by Eq. (1) using the Lyapunov method combined with the linear matrix inequality (LMI) technique. Unlike [23], in this paper the transformation model is only partly used for the getting criteria. Our results are approximately the same conservative as the existing ones in literature [23], but have much simpler mathematical presentation. Namely, our stability condition is expressed by means of one LMI, while the stability condition in [23] is presented by two LMI.

**Theorem 1.** System given by Eq. (1) is asymptotically stable for all \( h \) satisfying Inequality (3) if there exist real symmetric matrices \( P > 0, Q > 0 \) and \( Z > 0 \) satisfying the following LMI:

\[
    \begin{align*}
        \Gamma &\succ 0 \\
        \Gamma &\succ -Q \\
        \Gamma &\succ -Z
    \end{align*}
\]

where \( \Gamma \) is given by:

\[
    \begin{bmatrix}
        (A + B)^T PB & -A^T PB & h_u (A - I)^T Z \\
        * & -Q & -B^T PB & h_u B^T Z \\
        * & * & -Z & 0 \\
        * & * & * & -Z
    \end{bmatrix}
\]

\[
    \Omega < 0
\]

**Proof.** Let

\[
    \gamma = -P - P^T (h_m - h_m + 1)Q + (A - I)^T P (A + B) + (A + B)^T PA
\]
\[ \eta(m) \triangleq x(m+1) - x(m) \]
\[ = Ax(m) + Bx(m-h(m)) - x(m) \]
\[ = (A-I)x(m) + Bx(m-h(m)) \]
\[ \text{and} \]
\[ x(k-h(k)) = x(k) - x(k-h(k)) \]
\[ = x(k) - \sum_{m=k-h(k)}^{k-1} (x(m+1) - x(m)) \]
\[ = x(k) - \sum_{m=k-h(k)}^{k-1} \eta(m) \]

Then, the system given by Eq. (1) can be transformed into:
\[ x(k+1) = Ax(k) + Bx(k-h(k)) \]
\[ = Ax(k) + B \left( x(k) - \sum_{m=k-h(k)}^{k-1} \eta(m) \right) \]
i.e.
\[ x(k+1) = (A+B)x(k) - B \sum_{m=k-h(k)}^{k-1} \eta(m) \]

Choose a discrete-time counterpart Lyapunov-Krasovskii functional candidate as:
\[ V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) \]
\[ V_1(k) = x^T(k)Px(k) \]
\[ V_2(k) = \sum_{i=k-h(k)}^{k-1} x^T(i)Qx(i) \]
\[ V_3(k) = \sum_{i=k+1-h(k)}^{k-1} \sum_{j=k+1-i}^{i-1} x^T(i)Qx(i) \]
\[ V_4(k) = \sum_{i=k-h(k)}^{k-1} \sum_{j=k+1-i}^{i-1} \eta^T(j)Z\eta(j) \]

where \( P \), \( Q \) and \( Z \) are positive definite matrices to be determined.

Taking the forward difference:
\[ \Delta V = V(k+1) - V(k) \]
along the solutions of system given by Eq. (1) and transformed system given by Eq. (12), we can obtain:

\[ \Delta V_1(k) = x^T(k)Px(k+1) - x^T(k)Px(k) = [Ax(k) + Bx(k-h(k))]^T P(A+B)x(k) - B \sum_{m=k-h(k)}^{k-1} \eta(m) \]
\[ - x^T(k)Px(k) = x^T(k) \left[ A^TP(A+B) - P \right] x(k) + x^T(k) \left[ -A^TPB \right] \sum_{m=k-h(k)}^{k-1} \eta(m) + x^T(k-h(k))B^TP(A+B)x(k) + \sum_{m=k-h(k)}^{k-1} \eta(m) \]

Whereas:
\[ h(k+1) \leq h_m, \quad h(k) \geq h_m \]

then:
\[ \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) \leq \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) \]

Therefore:
\[ \Delta V_2(k) = x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \]
\[ + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) \]

\[ \leq x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \]
\[ + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) \]

\[ \Delta V_3(k) = x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \]
\[ + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) + \sum_{i=k+1-h(k)}^{k-1} x^T(i)Qx(i) \]
In addition:

\[ \Delta V_2(k) = \sum_{j=\beta_h}^{k-1} \sum_{i=k+j}^{\beta_h-1} x^T(i)Qx(i) - \\
- \sum_{j=\beta_h}^{k-1} \sum_{i=k+j}^{\beta_h-1} x^T(i)Qx(i) = \\
= \sum_{j=\beta_h}^{k-1} \sum_{i=k+j}^{\beta_h-1} \left[x^T(k)Qx(k) - \\
- x^T(k-1)Qx(k-1) \right] = \left(-h_m+1-(-h_m+2)\right)x^T(k)Qx(k) - \\
- \sum_{j=\beta_h}^{k-1} x^T(k-1)Qx(k-1) = \\
= (h_m-h_m)x^T(k)Qx(k) - \sum_{i=k-h_m}^{k-1} x^T(i)Qx(i) \]

Finally:

\[ \Delta V_4(k) = \sum_{i=k-h_m}^{k-1} \sum_{j=k+i}^{\beta_h-1} \eta^T(j)Z\eta(j) - \\
- \sum_{i=k-h_m}^{k-1} \sum_{j=k+i}^{\beta_h-1} \eta^T(j)Z\eta(j) = \\
= \sum_{i=k-h_m}^{k-1} \left(\eta^T(k)Z\eta(k) - \eta^T(k+i)Z\eta(k+i) \right) \\
= h_m\eta^T(k)Z\eta(k) - \sum_{i=k-h_m}^{k-1} \eta^T(k+i)Z\eta(k+i) \\
= h_m\eta^T(k)Z\eta(k) - \sum_{m=k-h_m}^{k-1} \eta^T(m)Z\eta(m) \]

From Eqs. (14) and (18)-(20), we have:

\[ \Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \leq x^T(k)\left[ A^T P(A+B) - P + \\
+ (d_u - d_m + 1)Q \right] x(k) + \\
+ x^T(k-h(k))B^T P(A+B)x(k) + \\
+ x^T(k-h(k))[-B^T PB + h_u(A-L)^T Z(A-L)x(k)] + \\
+ 2x^T(k)\left[ \frac{1}{2}A^T PB + h_u(A-L)^T ZB \right] x(k) - \\
- x^T(k-h(k))Qx(k-h(k)) + \sum_{i=k-h_m}^{k-1} x^T(i)Qx(i) + \\
+ (h_u-h_u)\eta^T(k)Z\eta(k) - \sum_{i=k-h_m}^{k-1} \eta^T(i)Qx(i) + \\
+ h_u\eta^T(k)Z\eta(k) - \sum_{m=k-h_m}^{k-1} \eta^T(m)Z\eta(m) \]

i.e.,

\[ \Delta V(k) \leq \xi^T(k)\Omega\xi(k) \]

where:

\[ \xi(k) = \left[ x^T(k) x^T(k-h(k)) \sum_{m=k-h_m}^{k-1} \eta^T(m) \right] \]

\[ \hat{\Omega} = \left[ \begin{array}{ccc} \frac{1}{2} \left( A^T PB + h_u(A-L)^T ZB \right) & -\frac{1}{2}A^T PB \\
-\frac{1}{2}B^T PB & -\frac{1}{2}B^T PB \end{array} \right] \]

\[ \hat{\gamma} = A^T P(A+B) - P + (h_u-h_m+1)Q + \\
+ h_u(A-L)^T Z(A-L) \]
Obviously, $\Delta V(k)<0$ if $\Omega<0$. Introducing a substitution $P \leftrightarrow 2P$, we have:

$$\hat{\Omega} = \begin{bmatrix}
\hat{\Gamma} (A+B)^{T} PB + h_u (A-I)^{T} ZB & -A^{T} PB \\
* & -Q + h_u B^{2} ZB & -B^{2} PB \\
* & * & -1/Z \\
\end{bmatrix} < 0$$

(27)

Further:

$$\hat{\Gamma} = -P - P^{T} + A^{T} PA + (h_u - h_u + 1)Q + h_u (A-I)^{T} h_u (A-I)^{T} Z(A-I)$$

Introducing a substitution $h_u Z \leftrightarrow Z$, we obtain the LMI given by Inequality (9). From this, it follows that the Inequality (9) guarantees that $\Delta V(k)<0$ for all non-zero $\zeta(k)$. Hence, the system given by Eq. (1) is asymptotically stable for all time-varying delay $h(k)$ satisfying Inequality (3).

**Remark 1.** The condition given by Inequality (9) in Theorem 1 is LMI condition, therefore can be easily checked by using standard numerical software. This condition depends on both the maximum and minimum delay bounds. In [23] the stability condition is defined by two LMI and numerically more complex.

**Remark 2.** The Lyapunov-Krasovskii functionals defined in this work (Theorem 1) and [23] (Theorem 1) are mutually identical. However, they differ with respect to the forward difference $\Delta V = V(k+1) - V(k)$. For performing the forward difference [23] in a model transformation given in Eq. (12) is used twice, whereas in our paper used only once (see Eq. (14)).

**Remark 3.** From Theorem 1, for the constant delay case ($h_u = h = h$), we have the next result.

**Corollary 1.** System given by Eq. (1) is asymptotically stable with $h(k) = h$ if there exist real symmetric matrices $P > 0, Q > 0$ and $Z > 0$ satisfying the following LMI:

$$\Phi = \begin{bmatrix}
T & (A+B)^{T} PB & h(A-I)^{T} Z \\
* & -Q & hB^{2} Z \\
* & * & -Z \\
\end{bmatrix} < 0$$

(31)

Proof. From the condition given by Inequality (9), for $h_u = h = h$, follows the condition given by Inequality (31).

**RESULTS AND DISCUSSION**

In this section, one example is used to demonstrate that the method presented in this paper is effective in implementation.

**Example 1.** Consider the following two discrete-time systems with time-varying delay in the state:

**S1:**

$$x(k+1) = \begin{bmatrix}
0.8 & 0 \\
0.05 & 0.9 \\
\end{bmatrix} x(k) + \begin{bmatrix}
-0.2 & 0 \\
-0.4 & -0.1 \\
\end{bmatrix} x(k-h(k)),$$

$$x(k) = [x_1(k) \ x_2(k)]^{T}$$

**S2:**

$$x(k+1) = \begin{bmatrix}
0.6 & 0 \\
0.35 & 0.7 \\
\end{bmatrix} x(k) + \begin{bmatrix}
0.1 & 0 \\
0.2 & 0.1 \\
\end{bmatrix} x(k-h(k)),$$

$$x(k) = [x_1(k) \ x_2(k)]^{T}$$
Let first determine the upper limit of delay $h_m$ for $h_m \in \{0, 1, 2, 3\}$, so that the systems S1 and S2 are asymptotically stable. Using Theorem 1 and Theorem 1 from [23], the upper limit of delay $h_m$ for given $h_m$ are calculated and shown in Table 1. It is clear that the obtained results are almost identical. However, the advantage of our results compared to the results presented in [23] is the simpler mathematical and numerical form. Namely, in Theorem 1 the stability condition is expressed by means of one LMI, while the stability condition (Theorem 1 in [23]) is presented by two LMI.

Also, using Corollary 1, the upper limit of constant delay $h$ is calculated and shown in Table 1, so that the systems S1 and S2 are yet stable. It is obvious that Corollary 1 gives wider range of delay. This is because the Corollary 1 assumes that the delay is constant, as opposed to Theorem 1 which takes into account the time-varying delay.

In order to verify previous results, the system operation is simulated under the following conditions:

S1:
- time-delay: a) $h(k) = h = 5$ or b) $2 \leq h(k) \leq 5$ is random integer variable.
- Initial state:
  $$x(\theta) = \psi(\theta) = \begin{bmatrix} 3 & 4 & 5 & 5 & 4 & 3 \\ -3 & -3 & -3 & -3 & -3 & -3 \end{bmatrix}^T,$$
  $$\theta \in \{-5, -4, \ldots, 0\}$$

S2:
- time-delay: a) $h(k) = h = 10$ or b) $2 \leq h(k) \leq 10$ is random integer variable.
- Initial state:
  $$x(\theta) = \psi(\theta) = \begin{bmatrix} 5 & 4 & 3 & 2 & 2 & 3 & 4 & 5 & 5 & 4 & 3 \\ -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 & -3 \end{bmatrix}^T,$$
  $$\theta \in \{-10, -9, \ldots, 0\}$$

Figures 1 and 2 show the initial condition response of the systems S1 with $h(k) = h = 5$ and S2 with $h(k) = h = 10$. It is observed that the values of

Table 1. Upper bounds of time delay

<table>
<thead>
<tr>
<th>System</th>
<th>Theorem 1 [23]</th>
<th>Theorem 1, this work</th>
<th>Corollary 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>$0 \leq h(\theta) \leq 4$</td>
<td>$0 \leq h(\theta) \leq 4$</td>
<td>$h \leq 11$</td>
</tr>
<tr>
<td></td>
<td>$1 \leq h(\theta) \leq 4$</td>
<td>$1 \leq h(\theta) \leq 4$</td>
<td>$h \leq 11$</td>
</tr>
<tr>
<td></td>
<td>$2 \leq h(\theta) \leq 3$</td>
<td>$2 \leq h(\theta) \leq 5$</td>
<td>$h \leq 19$</td>
</tr>
<tr>
<td></td>
<td>$3 \leq h(\theta) \leq 5$</td>
<td>$3 \leq h(\theta) \leq 5$</td>
<td>$h \leq 19$</td>
</tr>
<tr>
<td>S2</td>
<td>$0 \leq h(\theta) \leq 10$</td>
<td>$0 \leq h(\theta) \leq 10$</td>
<td>$h \leq 19$</td>
</tr>
<tr>
<td></td>
<td>$1 \leq h(\theta) \leq 11$</td>
<td>$1 \leq h(\theta) \leq 11$</td>
<td>$h \leq 19$</td>
</tr>
<tr>
<td></td>
<td>$2 \leq h(\theta) \leq 12$</td>
<td>$2 \leq h(\theta) \leq 11$</td>
<td>$h \leq 19$</td>
</tr>
<tr>
<td></td>
<td>$3 \leq h(\theta) \leq 13$</td>
<td>$3 \leq h(\theta) \leq 11$</td>
<td>$h \leq 19$</td>
</tr>
</tbody>
</table>

Figure 1. Initial condition response of the system S1 with constant time-delay $h(k) = h = 5$.

Figure 2. Initial condition response of the system S2 with constant time-delay $h(k) = h = 10$. 
state variables $x \to 0$ when $t \to \infty$, which proves that the systems $S_1$ and $S_2$ are asymptotically stable.

The initial condition response of the systems $S_1$ with time-varying delay and sequence time-varying delay $2 \leq h(k) \leq 5$ as random integer variable are shown in Figures 3 and 4. From Figure 3, it is also observed that when $t \to \infty$, the values of state variables $x \to 0$, which proves that the system $S_1$ is asymptotically stable.

Figures 5 and 6 show the initial condition response of the systems $S_2$ with time-varying delay and sequence time-varying delay $2 \leq h(k) \leq 10$ as random integer variable. From Figure 5, it can be also concluded that the system $S_2$ is asymptotically stable.

CONCLUSION

In this paper, the problem of delay-dependent stability for discrete time systems with time-varying state delay is investigated. Some new LMI sufficient conditions are proposed, which is dependent of the minimum and maximum delay bounds. This stability condition can be easily verified via standard numerical software. It was shown that the results coincide with the existing results from the literature. The advantages of our results, compared to the previously published results, are the simpler mathematical form and the numerical efficiency. Namely, in Theorem 1, we express the stability condition by means of one LMI, while previously two LMIs have been employed. Finally, a numerical example is included to show that our results are approximately the same conservative as the existing ones in the literature.
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REFERENCES