OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR A THIN FILM FLOW OF A PSEUDO-PLASTIC FLUID DRAINING DOWN OR LIFTING UP ON A CYLINDRICAL SURFACE

In this study, the pseudo-plastic model is used to obtain a solution for steady thin film flow on the outer surface of a long vertical cylinder for lifting and drainage problems. The non-linear governing equations subject to appropriate boundary conditions are solved analytically for velocity profiles by a modified homotopy perturbation method called the optimal homotopy asymptotic method. Expressions for the velocity profile, volume flux, average velocity, shear stress on the cylinder, normal stress differences, and force to hold the vertical cylindrical surface in position have been derived for both the problems. For the non-Newtonian parameter \( \beta = 0 \), we retrieve Newtonian cases for both the problems. We also plot and discuss the effects of the Stokes number, \( S_t \), the non-Newtonian parameter, \( \beta \), and the thickness, \( \delta \), of the fluid film on the fluid velocities.

Keywords: Lifting and drainage problems, pseudo plastic fluid, optimal homotopy asymptotic method.
contact with stationary air. Our aim here is to find the effects of the non-Newtonian parameter $\beta$ on the steady flow of pseudo plastic fluid flowing down on an infinite vertical cylinder.

Recently, thin film flows of non-Newtonian fluids have become an attractive research field for due to their wide applications in nonlinear sciences and engineering industries. Landau et al. [4] discussed the drainage thin film flow of Newtonian fluids. Siddiqui et al. [5] found the results for a drainage problem for thin film flow of a fourth grade fluid down a vertical cylinder and found the exact solution for the Phan-Thein-Tanner (PTT) fluid for both lifting and drainage problems [6]. Siddiqui et al. [7] also considered the same flow of non-Newtonian fluids on a moving belt.

The optimal homotopy asymptotic method was first proposed by Marinca and Herisanu in 2008 [10]. It is a modified homotopy perturbation method with the help of the least squares technology [11]. The method was re-named as the optimal homotopy-analysis approach [15].

Recently, Marinca and Herisanu [10-13] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. In their work they have used this method to understand the behavior of nonlinear mechanical vibration of electrical machine. They also used the same method for the solution of nonlinear equations arising in the steady state flow of a fourth-grade fluid past a porous plate and for the solution of nonlinear equation arising in heat transfer. Here we use OHAM for the solution of the thin film flow problems and it is observed that OHAM is an easy, flexible and reliable method.

The purpose of this paper is to discuss the nonlinear problems of thin film flow on cylindrical surfaces that arise from the mathematical modeling of pseudo plastic fluid In the present paper, we consider the thin film flow of an incompressible, pseudo plastic fluid on the surface of a vertical cylinder for both lifting and drainage problems. To the best of the authors’ knowledge, no attempt has been made so far to discuss the thin film flow of a pseudo plastic fluid on a cylindrical surface, and this type of problem is solved by OHAM for the first time.

In this paper, the governing equations, formulation of the drainage and lifting problems are developed. The basics of the Optimal Homotopy Asymptotic method, solution of the problem, average velocities and volume flow rates are given for both the problems, and the obtained results are discussed.

**Governing equations**

The basic equations governing the motion of an isothermal, homogeneous, incompressible fluid are:

$$\text{div } \mathbf{V} = 0 \quad (1)$$

$$\rho \frac{D \mathbf{V}}{Dt} = \rho \mathbf{f} - \text{grad } P + \text{div } \mathbf{S} \quad (2)$$

where $\rho$ is the constant fluid density, $\mathbf{V}$ is the velocity vector, $\mathbf{f}$ is the body force per unit mass, $P$ denotes the dynamics pressure, $Dt$ denotes the material derivative defined as:

$$\frac{D(\cdot)}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla)(\cdot)$$

$\mathbf{S}$ is the extra stress tensor which for pseudo plastic fluid model is defined as [8]:

$$\mathbf{S} + \lambda \mathbf{S} + \frac{1}{2}(\lambda - \mu_i)(\mathbf{A}_i \mathbf{S} + \mathbf{S} \mathbf{A}_i) = \eta_i \mathbf{A}_i \quad (3)$$

where $\eta_i$ is the zero shear viscosity, $\lambda_i$ is the relaxation time and $\mu_i$ is the material constant.

The first Rivlin-Ericksen tensor, $\mathbf{A}_i$, is defined as:

$$\mathbf{A}_i = (\text{grad } \mathbf{V}) + (\mathbf{grad } \mathbf{V})^\top \quad (4)$$

The contravariant convected derivative denoted by super imposed $\nabla$ over $\mathbf{S}$ is defined as:

$$\nabla = \frac{DS}{Dt} \quad \{ (\mathbf{grad } \mathbf{V})^\top \mathbf{S} + \mathbf{S} (\mathbf{grad } \mathbf{V}) \} \quad (5)$$

It can be noted that if we substitute $\lambda_i = \mu_i = 0$ in Eq. (3), we recover the constitutive equation of a Newtonian fluid.

**Formulation of the drainage problem**

Consider a non-Newtonian, incompressible, laminar pseudo-plastic fluid falling on the outside surface of an infinitely long vertical cylinder of radius $R$, in the form of a thin, uniform axisymmetric film of thickness $\delta$, in contact with stationary air (Figure 1). The flow is driven through the action of gravity alone and the ambient air is assumed to be stationary.

The flow is assumed to be steady and the surface tension of the fluid negligible, and that the pressure is atmospheric. In view of the geometry involved we use cylindrical coordinates $(r, \theta, z)$ and consider the $\theta$ direction is normal to the cylinder and the $z$-direction is along the cylinder which is in downward direction. We assume the velocity vector $\mathbf{V}$ and the stress tensor $\mathbf{S}$ as:

$$\mathbf{V} = [0, 0, \omega(r)] \quad (6)$$
The boundary conditions for the problem are:

\[ S_\alpha = 0 \quad \text{at} \quad r = R + \delta \quad \text{(free surface)} \quad (8) \]
\[ w = 0 \quad \text{at} \quad r = R \quad \text{(no slip condition)} \quad (9) \]

where \( S_\alpha \) is the shear stress component of pseudo plastic fluid. By substituting Eq. (6) in Eqs. (1) and (2), the continuity Eq. (1) is identically satisfied and the momentum [2] of Eq. (2) reduces to:

\[
0 = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{d}{dr}(rS_\alpha) + \rho f_1
\]
\[
0 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r} \frac{d}{dr}(r^2 S_\alpha) + \rho f_2
\]
\[
0 = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{d}{dr}(rS_\alpha) + \rho f_3
\]

where \( f_1, f_2 \) and \( f_3 \) are components of body force in \( r, \theta \) and \( z \) directions, respectively.

\[
\begin{align*}
\mathbf{S} &= \mathbf{S}(r) \\
S_\alpha &= \eta_0 \left( \frac{d w}{dr} \right) \frac{1}{1 + \left( \lambda_1^2 - \mu_1^2 \right)} \left( \frac{d w}{dr} \right)^2
\end{align*}
\]

\[
S_{r} = \frac{\eta_0 \left( \frac{d w}{dr} \right)}{1 + \left( \lambda_1^2 - \mu_1^2 \right)} \left( \frac{d w}{dr} \right)^2
\]

On substituting the value of \( S_\alpha \), Eq. (11) reduces to the following form:

\[
0 = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{d}{dr}(rS_r) + \rho f_r
\]

and the boundary conditions (8) and (9) into:

\[
\frac{d w}{dr} = 0 \quad \text{at} \quad r = R + \delta \quad \text{(free surface)} \quad (14)
\]
\[ w = 0 \quad \text{at} \quad r = R \quad \text{(no slip condition)} \quad (15) \]

Introducing dimensionless parameters as:

\[
\beta = \frac{U_0}{\eta \delta}, \quad \delta^* = \frac{\delta}{R}, \quad S_r^* = \frac{S_r}{\eta_0 \frac{U_0}{\delta}}
\]

The dimensionless form of Eq. (13) and boundary conditions (14) and (15) by omitting the “*” become:

\[
0 = -\beta \frac{d w}{d r}
\]

where \( \beta = \frac{\delta^*}{1 + \delta^*} \) and \( S_r = \rho g \delta^2 \eta_0 U_0 R \) represents the Stokes number. Integrating Eq. (22) once with respect to \( r \) and using boundary condition (23), we get:

\[
r \frac{d w}{d r} - S_r \frac{\beta \left( (1 + \delta^2)^2 - r^2 \right)}{2} \left( \frac{d w}{d r} \right)^2 = S_r \left( (1 + \delta^2)^2 - r^2 \right)
\]

\[
S_r = \frac{S_r}{\left( (1 + \delta^2)^2 - r^2 \right)}
\]
\[ w = 0 \text{ at } r = 1 \]

or:

\[ S_{r} = \frac{S}{2r} \left(1 + \delta^2 - r^2\right) \]  \hspace{1cm} (19)

\[ w = 0 \text{ at } r = 1 \]

The above Eq. (18) along with one boundary condition is a highly non-linear first order differential equation. It is to be noted that this problem is a well-posed problem but it is difficult to find its exact solution; thus, we use optimal homotopy asymptotic method (OHAM) to solve this problem.

**Basic idea of optimal homotopy asymptotic method (OHAM)**

In this section, we introduce the basic idea of optimal homotopy asymptotic method (OHAM) [9-11] and solve Eq. (18) using this method.

We apply OHAM to the following differential equation:

\[ L(w(r)) + N(w(r)) + g(r) = 0, \quad B(w) = 0 \]  \hspace{1cm} (20)

where \( L \) is a linear operator, \( r \) denotes an independent variable, \( w(r) \) is an unknown function, \( g(r) \) is a known function, \( N(w(r)) \) is a nonlinear operator and \( B \) is a boundary operator. The OHAM now constructs an optimal homotopy \( \psi(r,p) : B \times [0,1] \rightarrow \mathbb{R} \) which satisfies the following homotopy equation, see for example [9-11].

\[ (1 - p)[L(\psi(r,p)) + g(r)] = H(p)[L(\psi(r,p)) + g(r) + N(\psi(r,p))], \quad B(\psi(r,p)) = 0 \]  \hspace{1cm} (21)

where \( p \in [0,1] \) is an embedding parameter, \( H(p) \) is a nonzero auxiliary function for \( p \neq 0 \) and \( H(0) = 0 \). \( \psi(r,p) \) is an unknown function. Obviously, when \( p = 0 \) and \( p = 1 \), it holds that:

\[ \psi(r,0) = w_0(r), \quad \psi(r,1) = w(r) \]  \hspace{1cm} (22)

Thus, as \( p \) increases from 0 to 1, the solution \( \psi(r,p) \) varies from the initial guess \( w_0(r) \) to the solution \( w(r) \), where \( w_0(r) \) is obtained from equation (21) for \( p = 0 \):

\[ L(w_0(r)) + g(r) = 0, \quad B(w_0) = 0 \]  \hspace{1cm} (23)

We choose the auxiliary function \( H(p) \) in the form:

\[ H(p) = \rho C_1 + \rho^2 C_2 + ... \]  \hspace{1cm} (24)

or we also choose the auxiliary function in the form [12,13]:

\[ H(r,p) = \rho h_1(r,C_1) + \rho^2 h_2(r,C_1) \]

where \( C_1, C_2, ... \) are auxiliary constants to be determined in an optimal manner.

Next the method expands \( \psi(r,p,C_1) \) into a Taylor’s series about the parameter \( p \) as follows [10]:

\[ \psi(r,p,C_1) = w_0(r) + \sum_{i=1}^{M} w_i(r,C_1) p^i, \quad i = 1, 2, ..., M \]  \hspace{1cm} (25)

Convergence of the series (25) depends upon the auxiliary constants \( C_1, C_2, ... \), and if it converges at \( p = 1 \) then we have [10]:

\[ \psi(r,C_1) = w_0(r) + \sum_{i=1}^{M} w_i(r,C_1), \quad i = 1, 2, ..., M \]  \hspace{1cm} (26)

Substituting Eq. (25) into Eq. (24) and equating the like powers of \( p \), the original nonlinear problem is converted into a sequence of linear problems. The resulting linear problems can now be solved and their solutions are used to construct an \( M^\text{th} \) order solution, which involves \( C_1 \), of the original problem through Eq. (26). Then inserting Eq. (24) into Eq. (21) results in the following residual:

\[ R(r,C_1) = L(w^{(m)}(r,C_1)) + g(r) + N(w^{(m)}(r,C_1)) \]  \hspace{1cm} (27)

If \( R(r,C_1) = 0 \), for some values of \( C_1 \), then \( w^{(m)}(r,C_1) \) will coincide with the exact solution.

However, this does not happen in general, especially in nonlinear problems. Therefore, optimal values of the auxiliary constants \( C_1, C_2, ..., C_M \) are calculated aimed at minimizing the following functional \( J_{10} \):

\[ J(C_1, C_2, ..., C_m) = \int_0^1 R^2(r,C_1, C_2, ..., C_m) dr \]  \hspace{1cm} (28)

Thus, the unknown constants \( C_i (i = 1, 2, ..., m) \) can be optimally identified from the following conditions [10]:

\[ \frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = ... = 0 \]  \hspace{1cm} (29)

With these known values of the auxiliary constants, \( C_1, C_2, ..., C_M \), the approximate solution (34) is now well determined.

**Solution of the drainage problem by OHAM**

In this section, we find the velocity profile for the non-linear first order differential Eq. (18) for drainage problem by using the newly introduced analytical method OHAM [9-11]. We construct a homotopy according to Eq. (21) for the non-linear differential Eq. (18) along with the boundary condition. Using the given values in the homotopy we obtain the order problems with the corresponding boundary conditions as follows.
i) Zeroth-order problem

The zeroth-order problem is given by:

\[
\frac{d}{dr} \left( \frac{1}{2} \left( r^2 - (1 + \delta)^2 \right) \right) + \frac{S_r}{2} \left[ r^2 - (1 + \delta)^2 \right] = 0
\]  

(30)

subject to boundary condition:

\[ w_0 = 0 \quad \text{at} \quad r = 1 \]

ii) First-order problem

The first-order problem is defined as follows:

\[
\frac{d}{dr} \left( \frac{1}{2} \left( r^2 - (1 + \delta)^2 \right) \right) + \frac{S_r}{2} \left[ r^2 - (1 + \delta)^2 \right] - \frac{d}{dr} \left[ \frac{\beta C S_r}{2} \left( \frac{d}{dr} w_0 \right) \right] \left[ \left( 1 + \delta \right)^2 - r^2 \right] = 0
\]

(31)

along with the boundary condition:

\[ w_1 = 0 \quad \text{at} \quad r = 1 \]

iii) Second-order problem

We introduce the second-order problem:

\[
\frac{d}{dr} \left( \frac{1}{2} \left( r^2 - (1 + \delta)^2 \right) \right) + \frac{C_r S_r}{2} \left( \frac{d}{dr} \left( \frac{1}{2} \left( r^2 - (1 + \delta)^2 \right) \right) - 1 \right) +
\]

\[
+ \frac{d}{dr} \left[ \frac{\beta C S_r}{2} \left( \frac{d}{dr} w_0 \right) \right] \left[ \left( 1 + \delta \right)^2 - r^2 \right] - C_r - 1 = 0
\]

(32)

along with the boundary condition:

\[ w_2 = 0 \quad \text{at} \quad r = 1 \]

Solving Eqs. (30)-(32) with the corresponding boundary conditions, we obtain the solutions of zeroth-, first- and second-order problems as follows:

i) Zeroth-order solution

The solution of Eq. (30) is given by:

\[
w_0(r) = \frac{S_r}{2} \left( \frac{1}{2} \right) \left( r^2 - (1 + \delta)^2 \right) \ln r
\]

(33)

ii) First-order solution

The first order solution is obtained by solving equation (31) along with the boundary condition:

\[
w_1(r) = \frac{\beta S_r C_r}{192 r^2} \left[ (r^2 - 1)(r^4 - 2(1 + \delta)^6 -
\]

\[-r^2 \left( 5 + 6\delta(2 + \delta) \right) \right] + 12r^2 (1 + \delta)^4 \ln r
\]

(34)

iii) Second-order solution

Solving Eq. (32) subject to boundary condition, we get the second order solution:

\[
w_2(r) = \frac{\beta S_r C_r}{192 r^2} \left[ (r^2 - 1)(r^4 - 2(1 + \delta)^6 -
\]

\[-r^2 \left( 5 + 6\delta(2 + \delta) \right) \right] + 12r^2 (1 + \delta)^4 \ln r
\]

or, equivalently:

\[
w_{\text{drainage}}^{(2)}(r) = \frac{S_r}{2} \left( \frac{1}{2} \right) \left( r^2 - (1 + \delta)^2 \right) \ln r
\]

(35)
Here it should be noted that for $\beta = 0$, we get a solution for a Newtonian fluid [4].

Flow rate and average velocity of thin film flow of drainage problem

The flow rate per unit width is given by:

$$ Q = 2\pi \left[ \frac{1}{2} \int_1^{r^2} r \delta^{(2)}_{\text{drainage}} (r) dr \right] $$

Substituting Eq. (36) in above equation and then integrating, we obtain the flow rate of pseudo plastic fluids as:

$$ Q = \frac{2\pi S}{2} \left[ \left\{ \frac{(1+\delta)^3}{4} \left( 4 \left( \ln(1+\delta) - \frac{1}{2} \right) - (1+\delta)^6 + 1 \right) - \frac{1}{4} \right\} + \frac{BC_S}{16} \left( \frac{(1+\delta)^3}{6} + 2 \ln(1+\delta) - (1+\delta)^2 \right) - \frac{(1+\delta)^3}{4} \left( 48 \left( \frac{(1+\delta)^2}{2} \right) \ln(1+\delta) - \frac{(1+\delta)^2}{2} \right) + \left\{ 5 + 6\delta(2+\delta) \right\} - 2 \right] - \frac{1}{6} - \left( 1+\delta \right)^6 + \frac{1}{2} \left( 5 + 6\delta(2+\delta) \right) \right] + \left( \frac{1+\delta^2}{2} \left( 5 + 6\delta(2+\delta) \right) \right) \right] + \frac{1}{92} \left\{ 6\beta S \left( 2A(C_t + C_s) + 10\beta C_S S_i D \right) \ln(1+\delta) \right\} + 6\beta C_S S_i (1 + \frac{(1+\delta)^3}{2} - \frac{1}{2}) - \frac{D_x}{4} \left( \frac{(1+\delta)^3}{4} - \frac{1}{4} \right) + \left( \frac{(1+\delta)^6}{6} - \frac{1}{6} \right) \right\} $$

$$ + \beta^2 C_t S_i^2 \left\{ (1+\delta)^6 \right\} - \frac{6}{16} \left\{ (1+\delta)^6 \right\} - 2 \left( \frac{(1+\delta)^3}{3} - \frac{1}{3} \right) + (6D_x - D_y) \left[ \left( 1 + \delta \right) - 1 \right] $$

$$ - D_x \left( \frac{(1+\delta)^3}{4} - \frac{1}{4} \right) + \beta C_S S_i^2 (72D_x (1 + C_s)) + 12\beta C_S S_i^2 A \left( \frac{(1+\delta)^3}{4} \left( 2\ln(1+\delta) - 1 \right) + \frac{1}{4} \right) \right] $$

where $A, D_x, D_y, D_z, D_b, D_c$ and $D_y$ are constants which are given in Appendix.

The average velocity, $\bar{w}$, is given by:

$$ \bar{w} = \frac{Q}{\pi \left( 1 + \delta^2 \right)} $$

Normal stress difference

The expressions (12) show that the normal stress difference is given by:

$$ S_x - S_\omega = \frac{2\eta_s (\lambda - \mu_\omega) \left( \frac{dw}{dr} \right)^2}{2 + \left( \lambda_\omega^2 - \mu_\omega^2 \right) \left( \frac{dw}{dr} \right)^2} = 2 \left( \frac{dw}{dr} \right) S_x $$

which, with help of Eqs. (44) and (22), yields:

$$ S_x - S_\omega = \frac{S_i}{16r^4} \left\{ r^2 - (1+\delta)^2 \right\} \times \left[ 8r^4 S_x - 2r^2 \beta \left( r^2 - (1+\delta)^2 \right)^2 \right] \times \left( C_t (2 + C_s) + \beta^2 C_t S_i^5 \left( r^2 - (1+\delta)^2 \right)^4 \right) $$

The expression (38) reveals that the normal stress difference vanishes at the free surface $r = 1 + \delta$.

The shear stress on the cylinder

The expression for the shear stress on the vertical cylindrical surface is:

$$ S_x \bigg|_{r=1} = \frac{S_i \delta (2 + \delta) \left[ 8 + 8\beta S_i^2 \left( 2 + \delta \right)^2 \left( -2C_2 + C_1 \left( -4 + C_3 \left( -2 + 2\beta \delta^4 (2 + \delta)^3 S_i^3 \right) \right) \right) \right]}{16 \left( 1 + \frac{1}{256} \beta \left( 8\delta (2 + \delta) S_i - 2\beta \delta^3 (2 + \delta)^3 \left( C_t (2 + C_s) + C_s S_i^3 + \beta^2 \delta^4 (2 + \delta)^3 C_t S_i^5 \right) \right) \right) } $$

Force to hold the vertical cylindrical surface in Position

The force $F$ per unit width to hold the vertical cylinder surface in position can also be determined using the expression for shear stress at the cylinder surface:

$$ F = \frac{H}{W} \left( S_x \right)_{r=1} \, dr $$

where $H$ is the length of the cylinder. Using (19) and (39), we obtain:
Equation (40) can also be used to determine the length $H$ of the vertical cylinder, once the force per unit width is known.

In the following section, we revisit the lifting problem of the same fluid on an infinite vertical cylinder.

**Lifting problem for the pseudo-plastic fluid**

We consider a container filled with an incompressible non-Newtonian (pseudo-plastic) fluid as shown in Figure 2. Through this container, a cylinder moves vertically upward with a constant speed $U_0$. The cylinder picks up a thin film fluid of uniform thickness $\delta$. The gravity tries to make the fluid drain down the cylinder.

![Figure 2. Geometry of the problem.](image)

The governing Eqs (1) and (2), after using Eq. (6) become:

$$\frac{\rho}{2} \left( r \frac{d}{dr} \right) \left( r \frac{d}{dr} - \rho g \right) = \frac{1}{r} \frac{d}{dr} \left( S_r \right) - \rho g$$

Using the expression for $S_r$ from Eq. (17), we get:

$$\frac{d}{dr} \left[ \frac{\rho g}{1 + \left( \alpha^2 - \mu^2 \right)} \frac{d}{dr} \left( r \frac{d}{dr} \right) \right] = \rho g r$$

and the boundary conditions (8) and (9) into:

$$\frac{d}{dr} \left( r \frac{dw}{dr} \right) = \rho g r$$

subject to boundary condition:

$$w_\theta = U_0 \quad \text{at} \quad r = 1$$

Equation (44) along with one boundary condition is a highly non-linear first order differential equation. It is to be noted that this problem is a well-posed problem but it is difficult to find its exact solution; thus, we use the optimal homotopy asymptotic method (OHAM) \[10-12\] for the solution.

i) Zeroth-order problem

The zeroth-order problem is given by:

$$r \frac{dw_0}{dr} + \frac{S_r}{2} \left[ (1 + \delta^2)^2 - r^2 \right] = 0$$

subject to boundary condition:

$$w_\theta = U_0 \quad \text{at} \quad r = 1$$

ii) First-order problem

The first-order problem is defined as follows:
\[
\begin{align*}
\frac{d w_2}{d r} + \frac{S_1}{2} (r^2 - (1 + \delta)^2) + C_1 (r^2 - (1 + \delta)^2) - &+ r^4 \{ -20 - 3\delta^2 (2 + \delta)^2 \{ -30 + \\
\frac{d w_2}{d r} + \frac{C_1 S_2}{2} (r^2 - (1 + \delta)^2) + \frac{1}{2} \frac{d w_2}{d r} + \frac{\beta C S_r}{2} \frac{d w_2}{d r} (r^2 - (1 + \delta)^2) = 0
\end{align*}
\] (46)

along with the boundary condition:
\[
w_1 = 0 \text{ at } r = 1
\]

\section*{iii) Second-order problem}

We introduce the second order problem:
\[
\begin{align*}
\frac{d w_2}{d r} + \frac{C_1 S_2}{2} (r^2 - (1 + \delta)^2) + C_2 \frac{d w_2}{d r} \frac{\beta C S_r}{2} \frac{d w_2}{d r} (r^2 - (1 + \delta)^2) - &- (1 + \delta)^2) - 1] \frac{d w_1}{d r} \frac{\beta C S_r}{2} \frac{d w_2}{d r} (r^2 - (1 + \delta)^2) - C_1 - 1] = 0
\end{align*}
\] (47)

along with the boundary condition:
\[
w_2 = 0 \text{ at } r = 1
\]

\section*{1) Zeroth-order solution}

The solution of Eq. (45) is given by:
\[
w_0(r) = 1 + \frac{S_1}{2} \left( r^2 - 2 - (1 + \delta)^2 \ln r \right)
\] (48)

\section*{ii) First-order solution}

The first order solution is obtained by solving Eq. (46) along with the boundary condition:
\[
w_1(r) = - \frac{\beta C S_r^3}{32 r^2} [(r^2 - 1)(r^4 - 2(1 + \delta)^6) - r^2 (5 + 6\delta (2 + \delta))] + 12r^2 (1 + \delta)^4 \ln r
\] (49)

\section*{iii) Second-order solution}

Solving Eq. (47) subject to boundary condition, we get the second order solution:
\[
w_2(r) = \frac{\beta S_r^3 C}{192 r^2} \left[ -6r^2 [(r^2 - 1)(r^4 - 2(1 + \delta)^6) - \right.\left. r^2 (5 + 6\delta (2 + \delta))] + 12r^2 (1 + \delta)^4 \ln r \right]
\]

where
\[
2 + \frac{\beta S_r^3 C}{192 r^2} \left[ -6r^2 [(r^2 - 1)(r^4 - 2(1 + \delta)^6) - \right.\left. r^2 (5 + 6\delta (2 + \delta))] + 12r^2 (1 + \delta)^4 \ln r \right]
\]

Here it should be noted that for \( \beta = 0 \), we get a solution for a Newtonian fluid for the lifting case.

\section*{Flow rate and average velocity of thin film flow of drainage problem}

The flow rate per unit width is given by:
\[
Q = 2\pi \int_0^1 \overline{w_{\text{thirg}}}(r) dr
\] (52)
Substituting Eq. (51) in Eq. (52) and then integrating, we obtain the flow rate of pseudo plastic fluids as:

\[
Q = \left( \frac{(1+ \delta)^2}{2} - \frac{1}{2} \right) + \frac{\pi \delta}{2} \left[ \left( \frac{(1+ \delta)^2}{4} \right) \left( \ln(1+ \delta) - \frac{1}{2} \right) + (1+ \delta)^6 - 1 \right] - \frac{\beta C S}{16} \delta^2 \left( \frac{1}{6} + 2 \ln(1+ \delta) - (1+ \delta)^2 \right) - \frac{(1+ \delta)^4}{4} \left( 4(1+ \delta)^2 \ln(1+ \delta) - (1+ \delta)^2 \right) - \frac{1}{6} \delta + \frac{1}{2} (5+ 6 \delta(2+ \delta)) \right] + \frac{(1+ \delta)^2}{2} (5+ 6 \delta(2+ \delta)) + \frac{1}{2} + \frac{6 \beta S}{92} (2A(C^2 + C_1 + C_2) + 10 \beta C S D) \ln(1+ \delta) - 6 \beta C S \delta^2 (1+ C_1 + D_2) \left( \frac{(1+ \delta)^2}{2} - \frac{1}{2} \right) - \frac{D}{4} \left( \frac{(1+ \delta)^4}{4} - \frac{1}{4} \right) + \frac{(1+ \delta)^6}{6} - \frac{1}{6} \delta^2 C_2 S^2 (15(1+ \delta)^2 \frac{(1+ \delta)^6}{6} - \frac{1}{6} \delta) - \frac{2}{3} \left( \frac{(1+ \delta)^3}{3} - \frac{1}{3} \right) + (6D_1 - D_2) \left[ (1+ \delta) \right] - \frac{(1+ \delta)^4}{4} \left( \frac{(1+ \delta)^4}{4} - \frac{1}{4} \right) + \beta C S^2 D_2 (72D_2 + (1+ \delta)) + 12 \beta C S D \left( \frac{(1+ \delta)^2}{2} (2 \ln(1+ \delta) - 1) + \frac{1}{4} \right) \right]
\]

The average velocity, \( \bar{W} \), is given by:

\[
\bar{W} = \frac{Q}{\pi [(1+ \delta)^2 - 1]}
\]

Normal stress difference

The expressions (14)-(16) show that the normal stress difference is given by:

\[
S_n - S_\infty = 2 \left( \frac{dW}{dr} \right) \left( \frac{r - (1+ \delta)^2}{r} \right)
\]

or:

\[
S_n - S_\infty = \frac{S_n}{192} \left( \frac{r - (1+ \delta)^2}{r} \right)^2 + 48 (r^2 - 1 - 2(1+ \delta)^2 \ln r) S_n - \frac{S^3}{r^2} \beta \delta ((r^2 - 1)(r^4 - 2(1+ \delta)^2)} - r^4 (5+ 6 \delta(2+ \delta)) + 12r^2 (1+ \delta)^4 \ln r \left( C_1 + C_2 \right) + S^3 \beta \delta (2r^{10} - 15r^4 (1+ \delta)^2) + 60 \delta (1+ \delta)^4 - 30r^2 (1+ \delta)^8 + 3 (1+ \delta)^8 + r^4 \left[ -20 - 3 \delta^3 (2+ \delta)^2 \right] - 30 + \delta (2+ \delta)) - 120r^4 (1+ \delta)^6 \ln r C^2]
\]

The above expression reveals that the normal stress difference vanishes at the free surface \( r = 1+ \delta \).

Shear stress on the vertical cylinder

The shear stress on the vertical cylindrical surface is:

\[
S_\sigma \mid_\sigma = - \delta S_\sigma (2+ \delta) \left[ 8 + \beta S^2 \delta^2 (2+ \delta)^2 \right] + \left[ -2C_2 + C_4 \left( C_2 - 2 + \beta \delta^5 \delta (2+ \delta)^2 \right) \right] + 16 \left( 1+ \frac{\beta}{256} \delta (2+ \delta) \delta \left( 2+ \delta \right)^3 \left( C_1 + C_2 \right) + \left[ + C_3 \right] S^3 + \beta \delta^5 (2+ \delta)^5 \left( C^2 \right) \right]
\]

Force to hold the vertical cylindrical surface in position

The force \( F \) per unit width to hold the vertical cylinder in position can also be determined using the expression for shear stress at the cylinder surface:

\[
F = \int_1^H (S_\sigma) \mid_\sigma \, dr
\]

where \( H \) is the length of the cylinder. Using Eqs. (6) and (53), we obtain:

\[
F = \frac{S_n}{2} \left( (1+ \delta)^2 - 1 \right) (1 - H)
\]
Equation (54) can also be used to determine the length $H$ of the cylinder, once the force per unit width is known.

**DISCUSSION OF RESULTS**

The graphs for the fluid velocities $w_{\text{drainage}}^{(2)}(r)$ and $w_{\text{lifting}}^{(2)}(r)$ against $r$ are plotted for both drainage and lifting problems various values of Stokes number $S_r$, the non-Newtonian parameter $\beta$ and the thickness $\delta$ of the fluid film respectively, by selecting different values of the auxiliary constants $C_1$ and $C_2$. From Figure 3a, b and c, it is found that the Stokes number $S_r$, the non-Newtonian parameter $\beta$ and the thickness $\delta$ of the fluid film have a direct correlation with the fluid velocity for the drainage case by taking the auxiliary constants $C_1 = -1.4084694$ and $C_2 = -0.0976192$.

In the lifting case, for the same values of auxiliary constants $C_1 = 1.036774$ and $C_2 = -4.147522$, an inverse relation observed for the significant parameters $S_r$, $\beta$ and $\delta$ in Figure 4a, b and c, respectively. The fluid velocity $w(r)$ versus the $r$-axis for drainage and lifting problems are shown here, plotted for different parameters. For the drainage case in Figure 5a-c, it is observed that the fluid velocity increases by varying the values of $S_r$, $\beta$ and $\delta$ for different values of $C_1 = -1.036767$ and $C_2 = -0.000449$. Similarly, for the lifting problem in Figure 6a-c, it is found that the velocity of the pseudo plastic fluid decreases by increasing the values of involved parameters $S_r$, $\beta$ and $\delta$ by taking $C_1 = 1.036774$ and $C_2 = -4.147522$. In the same way, by considering Figure 7a-c for the drainage problem, it is found that the velocity of the fluid increases for the Stokes number $S_r$, the non-Newtonian parameter $\beta$ and for the thickness $\delta$ of the fluid film by taking the auxiliary constants $C_1 = -1.4084694$ and $C_2 = -0.0976192$.

In the same way, by considering Figure 8a-c for the drainage problem, it is found that the velocity of the fluid increases for the Stokes number $S_r$, the non-Newtonian parameter $\beta$ and for the thickness $\delta$ of the fluid film by taking the auxiliary constants $C_1 = -1.4084694$ and $C_2 = -0.0976192$. It is also observed that, in Figure 9 for the non-Newtonian parameter $\beta = 0$, we get the graphs of Newtonian fluids for both the problems. In Table 1, the absolute difference shows the non-Newtonian effect on the velocity profile for the values:

$S_r = 1; \beta = 0.01; \delta = 1; C_1 = -1.4084694; C_2 = -0.0976192$

![Figure 3. The effect of: a) Stokes number $S_r$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile $w_{\text{drainage}}^{(2)}(r)$ by taking the constants $C_1 = -1.4084694$ and $C_2 = -0.0976192$.](image)
Figure 4. The effect of: a) Stokes number $S_t$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile $W^{(2)}(r)$ by taking the constants $C_1 = -1.4084694$ and $C_2 = -0.0976192$.

Figure 5. The effect of: a) Stokes number $S_t$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile for drainage problem by taking the constants $C_1 = -1.0367670$ and $C_2 = -0.0004492$. 
Figure 6. The effect of: a) Stokes number $S_t$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile for the lifting problem by taking the constants $C_1 = -1.0367670$ and $C_2 = -0.0004492$.

Figure 7. The effect of: a) Stokes number $S_t$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile for drainage problem by taking the constants $C_1 = 1.0467774$ and $C_2 = -4.1877176$.
Figure 8. The effect of: a) Stokes number $S_t$, b) non-Newtonian parameter $\beta$ and c) thickness $\delta$ on the dimensionless velocity profile for the lifting problem by taking the constants $C_1 = 1.0467774$ and $C_2 = -4.1877176$.

Figure 9. The Newtonian behavior of the fluid velocity for the lifting and drainage cases by taking $S_t = 1; \delta = 1.4; C_1 = 1.04678; C_2 = -4.18772$ and the non-Newtonian parameter $\beta = 0$.

Table 1. Values for the velocity profile in the drainage case for the viscous (Newtonian) case when $S_t = 1; \beta = 0.01; \delta = 1; C_1 = -1.4084694$ and $C_2 = -0.0976192$

<table>
<thead>
<tr>
<th>$r$</th>
<th>4th-order approximation</th>
<th>Exact (drainage)</th>
<th>Absolute difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.000000</td>
<td>0.00000</td>
<td>0.00000000</td>
</tr>
<tr>
<td>1.1</td>
<td>0.140882</td>
<td>0.13812</td>
<td>0.00276207</td>
</tr>
<tr>
<td>1.2</td>
<td>0.259045</td>
<td>0.254643</td>
<td>0.00440172</td>
</tr>
<tr>
<td>1.3</td>
<td>0.357586</td>
<td>0.352229</td>
<td>0.00535722</td>
</tr>
<tr>
<td>1.4</td>
<td>0.43884</td>
<td>0.432944</td>
<td>0.00589517</td>
</tr>
<tr>
<td>1.5</td>
<td>0.504612</td>
<td>0.49843</td>
<td>0.00618175</td>
</tr>
<tr>
<td>1.6</td>
<td>0.556329</td>
<td>0.550007</td>
<td>0.00632187</td>
</tr>
</tbody>
</table>
**CONCLUSION**

In this work, we have used optimal homotopy asymptotic Method proposed by Marinca to find the solution of the flow problem governed by Eqs. (18) and (44). The non-linear governing equations subject to appropriate boundary conditions are solved analytically for velocity profiles by the newly introduced optimal homotopy asymptotic method (OHAM). Explicit expressions for the velocity profile, volume flux, average velocity, shear stress on the cylinder, normal stress differences, and force to hold the cylindrical surface in position are obtained for both the problems. The graphical representations of the velocity profiles of lifting and drainage problems were presented as well.

In this study, all the results obtained from OHAM are logically good and converge to the exact solution as the constant $C_s$ increases in the auxiliary function. This method provides us a suitable way to control the convergence of the series solution using the auxiliary constants ($C_s$) which are optimally determined. We hope that this method has a great potential to help researchers, scientists and engineers of several field to develop a new non-linear analytical technique in the absence of small or large parameters.

**Appendix**

\[
A = 1 + 6\delta + 15\delta^2 + 20\delta^3 + 15\delta^4 + 6\delta^5 + \delta^6
\]

\[
D = 1 + 8\delta + 28\delta^2 + 56\delta^3 + 70\delta^4 + 56\delta^5 + 28\delta^6 + 8\delta^7 + \delta^8
\]

\[
D_1 = -40 + 3\delta[ -470 + \delta( -30\delta + \\
+ \delta( -270 + \delta( 40 + \delta( 3\delta + 8(10 + \delta))) )) ]
\]

\[
D_2 = 3[360\delta^3 + \delta(10 + \delta(45 + \\
+ \delta(12 + \delta(210 + \delta(120 + \delta(45 + \delta(10 + \delta))) )) )) ]
\]

\[
D_3 = 1 + 4\delta + 6\delta^2 + 4\delta^3 + \delta^4
\]

\[
D_s = 3 - \delta^2 \left( 24 + 40\delta + 30\delta^2 + 12\delta^3 + 2 \right)
\]

\[
D_s = 6(1 + \delta)^2
\]

\[
D_s = 60\delta(2 + \delta)(2 + \delta(2 + \delta))
\]

**REFERENCES**


Ključne reči: strujanje naviše i naniže, pseudo-plastični fluid, optimalna homotopska asimptotska metoda.