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Introducing *Formalism* in Economics: The Growth Model of John von Neumann

Summary: The objective is to interpret John von Neumann's growth model as a decisive step of the forthcoming formalist revolution of the 1950s in economics. This model gave rise to an impressive variety of comments about its classical or neoclassical underpinnings. We go beyond this traditional criterion and interpret rather this model as the manifestation of von Neumann's involvement in the formalist programme of mathematician David Hilbert. We discuss the impact of Kurt Gödel's discoveries on this programme. We show that the growth model reflects the pragmatic turn of the formalist programme after Gödel and proposes the extension of modern axiomatisation to economics.

Key words: Von Neumann, Growth model, Formalist revolution, Mathematical formalism, Axiomatics.

JEL: A12, B23, B41, C02.

In the immediate post-War years, Mark Blaug (1999, 2003) identified the emergence of a new paradigm in economics, the so-called "formalist paradigm", which marked the arrival of the pre-eminence of (mathematical) form over (theoretical) content, and which is mostly characterised by the crucial importance economists give to a specific (non-constructive) kind of demonstration of existence of equilibrium. This revolution took shape in the 1950s and 1960s around the works of Arrow, Debreu, Patinkin, Solow, Dorfmann, Samuelson and Koopmans.

The objective of this paper is to interpret John von Neumann's growth model (1937) as a decisive step of this formalist revolution, and by doing so, contribute to the definition of the formalist paradigm in economics. The 1937 model, it will be argued, is the manifestation of von Neumann's involvement in the formalist programme of mathematician David Hilbert, and provides economists with the new mathematical tools and methodology that will characterise the emerging paradigm in economics.

The 1937 paper gave rise to an impressive variety of contrasting comments as far as the filiations (classical versus neoclassical) of the growth model are concerned, and constitutes one of those enigmas which historians of economic thought are so fond of. However, the identification of an economic formalist paradigm allows one to go beyond the traditional demarcation line between classical and neoclassical economics and challenges the legitimacy of such a criterion. The issue of the nature of the assumptions upon which the 1937 model is based becomes much less relevant

than that of the extent of the methodological innovation introduced by von Neumann, namely, the introduction of the modern axiomatic approach in economics.

The aim of the following sections is to elucidate this interpretation through a rational reconstruction of the epistemological approach adopted by von Neumann in the 1937 paper. The result of this reconstruction may be summarised in this way: *von Neumann gives here an economic interpretation to a specific formal system which he initially elaborated in his previous work of 1928 on game theory.* Each term here has a precise meaning: a "formal system" is composed of (1) a set of symbols, (2) a set of rules for transforming these symbols into formulae, (3) a set of rules for transforming the formulae, and (4) a reduced number of formulae representing the axioms of the system to be observed. By construction, a formal system has no semantic content and may take on different interpretations. A "model" is an interpretation that is given to a formal system. The clear-cut separation between syntax and semantics – between the formal aspects of the system and its various interpretations – is one of the most salient characteristics of modern axiomatics.

In order to prove that the scope of the 1937 model may be correctly grasped by understanding von Neumann's global epistemological approach, we will proceed as follows. It is first necessary to offer a brief overview of the growth model and of the controversy over the filiations (section 1); the variety of the comments is by itself an invitation to consider an alternative interpretation. We found such an alternative in von Neumann's involvement in the formalist Hilbertian programme so that the classical/neoclassical demarcation line may well be replaced by the formalist/non-formalist criterion, as Blaug (2003) and Nicola Giocoli (2003) suggest (section 2). The term "formalism" is ambiguous and requires further elucidation. In particular, the question of the impact of Gödel's discoveries on the formalist programme is of primary interest to us to the extent that, it will be argued, the 1937 paper is a manifestation of the pragmatic turn that Gödel lays on formalist mathematicians (section 3). We will then have all the elements to show that von Neumann's main achievement in his 1937 paper has been to propose to economists the substitution of the mechanical analogy with the mathematical analogy, as a result of his participation in the post-Gödelian mathematical formalist programme (section 4).

1. The 1937 Model and its Various Interpretations

In the 1937 article, von Neumann characterises the equilibrium configuration of an economy expanding at a uniform rate. In equilibrium, prices are constant, as are the quantity ratios between different goods. Several simplifying assumptions are introduced by von Neumann to make equilibrium possible: constant returns to scale; pure and perfect competition; unlimited quantities of goods available through the productive process (this applies to land and labour, no primary factors existing in the model); no savings from workers who are depicted as draft animals; and no consumption from producers who save the totality of their income.

Production is considered a temporal process (of length of one period) of transforming one set of goods into another; for reasons of simplicity and for ensuring the unity of the solution, von Neumann also had to make the assumption that each good entered the productive process of all goods, be it as input or output, and also in an

arbitrarily small proportion. The cost of production of one good depends on the value of the goods necessary for its production, plus the interest rate; the prices of goods correspond to their production costs, whatever the preferences of workers or producers are.

Solving this model allows identifying the following.

- Which, among the set of goods in the economy, are the free goods whose price must be fixed equal to zero, and what the prices are of the other non-free goods; free goods are goods whose produced quantity exceeds the quantity used in the production process in a proportion higher than the rate of growth of the economy. Introducing the free goods rule allowed von Neumann to avoid the occurrence of negative prices at equilibrium, and, from a mathematical point of view, transform the representation of the economy by introducing linear inequalities into the model;
- Which are the profitable production processes and which ones are non-profitable and will, therefore, not be implemented (a profitability rule which, like the free goods rule, leads to the use of linear inequalities in the model); the model allows the determination of the maximum intensity with which each profitable process will be implemented – that is, the produced quantities of each good, and, thus, given the constant returns to scale assumption, the growth rate of the economy;
- The dual symmetry of the model is one of its essential properties and manifests as follows;
- Solving the model may be interpreted on the one hand as a *problem of technological choice*: given the price vector, it is possible to determine the vector of the maximum possible produced quantities and the optimal growth rate, under the constraint of the free goods rule and given the impossibility of consuming more than is produced;
- Solving the model may also be interpreted on the other hand as a *problem of economic expansion*, which turns out to be the mirror image of the previous problem. It consists of determining the optimal price vector and interest rate which prevail, given the intensities of production processes, the efficiency rule, and the competitive constraint according to which no extra profits are allowed.

Von Neumann showed that an equilibrium solution exists, that it is unique, and that the interest rate of this configuration is equal to the growth rate. The proof of existence breaks with the traditional attempts of demonstrating the existence of a general equilibrium configuration consisting of counting the numbers of equations and unknowns. Such an approach did not constitute sufficient proof of existence, and, furthermore, the model was formalised in terms of inequalities (the free goods rule and the profitability rule) and thus required specific mathematical tools. The demonstration of existence provided by the author consisted in an extension of Broüwer's Fix Point Theorem and represented the first introduction of topological tools in economic analysis: von Neumann introduced a new function, ф (X, Y), which represents the ratio between the total incomes and the total costs, and demon-

strates the existence of a solution of the growth model, amounting to demonstrating the existence of a saddle point for function ϕ. Now, the existence of this saddle point is itself the consequence of von Neumann's demonstration of a fix point lemma. This demonstration is non-constructive in the sense that no method is provided for the determination of the fix point; with this kind of demonstration, equilibrium thus becomes a purely logical concept. Existence is demonstrated by showing that non-existence would involve a logical contradiction. As emphasised by Giocoli (2003, p. 8) and also Blaug (2003, p. 146), this kind of non-constructive proof (or "negative proof") allows a direct jump from the axioms of the model to its final outcome and accounts for the neglect of mainstream economists in the analysis of the economic process that leads to equilibrium.

With the notable exception of Harold W. Kuhn and Albert W. Tucker (1958) who provide an analysis of the mathematics of von Neumann's proof, economists in the 1950s and 1960s mainly concentrated their comments on the economic filiations of this model. In 1959, the Kaldor–Solow debate that unfolded during the Corfù Conference on Capital was the starting point of a long controversy over the interpretation of the 1937 model. Kaldor insisted upon the classical underpinnings of von Neumann's growth model, whereas Solow emphasised the possibility of integrating this model into the neoclassical framework. The arguments advanced by the two economists set the tone of future debate.

- Supporters of a classical interpretation insist on the heterodox nature of the assumptions on which the model is built. Kaldor, for instance, essentially based his position on von Neumann's assumption of infinite expansion of primary factors for, according to him, one of the defining features of mainstream economics is precisely the existence of a physical constraint on the available quantity of these resources. In the same way, Luigi Pasinetti (1977) stressed the circular character of the production process, whereas Heinz Kurz and Neri Salvadori (1993) insisted on its temporal dimension and on the proximity of certain of the model's characteristics with past contributions of classical authors, from Petty to Remak and von Bortkiewicz. It is worth remarking that according to this line of interpretation, and contrary to what is defended below, the nature of the mathematical techniques used in demonstrations does not constrain the theoretical nature of the model. Accordingly, von Neumann's model would offer proof that optimisation tools do not constitute a selective feature of neoclassical economics;

- Supporters of a neoclassical interpretation put to the fore more technical arguments to show that the model may be understood as a special case of the more general neoclassical framework. Such generalisations entail, among others, the introduction into the model of the intertemporal preferences of consumers (Edmont Malinvaud 1953), the consideration of labour as a primary factor constrained by an exogenous growth rate (Michio Morishima 1964), a relaxation of the assumption of circularity according to which each production process uses or produces a given quantity of each good produced in the preceding period (John G. Kemeny, Oskar Morgen-

stern, and Gerald L. Thompson 1956), etc. This interpretation consists ultimately in presenting the 1937 model as a crucial step in the construction of the neoclassical paradigm, starting from Léon Walras (through the formulation given by Gustav Cassel) and extending to the modern demonstration of existence by Kenneth Arrow, and Gérard Debreu.[1]

It is possible to appraise the relevance of the controversy over the filiations of von Neumann's model from different perspectives. If it were simply a question of situating the model either in the classical or the neoclassical camp, then the extent of the confrontation would be rather narrow and the relevance of the debate questionable. However, from an analytical viewpoint, the implications of this confrontation have turned out to be very significant for both sides: in the orthodox camp, von Neumann's growth model is at the roots of linear programming, the turnpike theorem of Dorfman, Samuelson and Solow, and of modern proofs of existence of general equilibrium; in the heterodox camp, the growth model is certainly an important source of the classical revival of the 1960s that followed the publication of Sraffa's book. For instance, Goodwin's limit cycle model formalises short-term economic fluctuations along the quasi-stationary long-term equilibrium trend of von Neumann;[2] Andras Brody (1970) starts from a simplified version (with no joint production) in matrix form of von Neumann's model in order to propose a mathematical rehabilitation of the labour theory of value.

The variety of the interpretations ultimately shows that von Neumann's growth model hardly fits into the traditional classical/neoclassical classification system. It is a characteristic of path-breaking contributions to upset the prevailing schemes. Interpreting the growth model in the light of the forthcoming formalist revolution of the 1950s means focusing on the nature of the mathematical innovations introduced by von Neumann in economics. These innovations may be appraised from different perspectives.

From a strictly technical viewpoint, von Neumann's contribution is easy to identify: it consists in the generalisation of Broüwer's Fix Point Theorem. The original title of the paper is explicit: "About a System of Economic Equations and a Generalization of Broüwer's Fix Point Theorem". In 1945, Kaldor, then editor of the *Review of Economic Studies*, asked von Neumann to modify his title to "A Model of General Economic Equilibrium". However, the first sentence of the article is evidence of the author's priority: "The subject of this paper is the solution of a typical economic equation system...", adding a little further on that "... the mathematical proof is possible only by means of a generalization of Broüwer's Fix Point Theorem i.e. by the use of very fundamental topological facts. This generalised Fix Point Theorem…is also interesting in itself" (von Neumann 1945/46, p. 29).

In order to reach this strictly-defined objective, he adopts a typical mathematical approach (Mohammed Dore 1989a) which consists of encompassing this problem

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[1] Cf. Roy E. Weintraub (1985), and Bruna Ingrao and Giorgio Israel (1990).
[2] However, according to Richard Goodwin (1989) the most important error he made during his career had been to not acknowledge in his day that von Neumann's 1937 paper was one of the most interesting and fertile contributions of the twentieth century, at the origins – among other things – of the modern theory of growth.

(the extension of Broüwer's Theorem) within a set of more general problems (solving a system representing a growth economy), the resolution of which allows a solution of the original problem. This idea is endorsed by the fact that the Minimax Theorem is an unnecessarily heavy tool to demonstrate the existence of an equilibrium solution of this economy: Nicholas Georgescu-Roegen (1951) provides a demonstration exclusively based on the properties of convexity and separation of hyper-plans, supporting the idea that the growth model represented to von Neumann only a specific support which allowed him to back up his mathematical results.

From a methodological perspective, the contribution of the 1937 model is much more complex to identify. It is the objective of this rational reconstruction to show that von Neumann's path-breaking contribution consisted of extending the standards of rigour of mathematical formalism to the community of economists. Discussion about the nature of the model's theoretical foundations is relegated to the background.

It is worth noting that the majority of the protagonists to the filiations debate make a point of mentioning the limitations of their comments, recognising to a certain extent that the field of economics does not represent the privileged field of investigation of the author: Tjalling C. Koopmans (1964, p. 356) declared along this line that despite the unquestionable theoretical advance provided by the 1937 growth model, the paper is rather poor economics; in the same way, David G. Champernowne (1945/46, p. 10) conceded that the author approached the question of existence as a mathematician, putting the emphasis on aspects of the problems distinct from those upon which an economist would have insisted; notice also the comment of Sukhamoy Chakravarty (1989, p. 70) who, before introducing the Kaldor–Solow debate, asserted that it was possible ultimately that von Neumann himself considered his paper as essentially technical in nature.[3]

"God, it is said, speaks to each of us in our own language…", Paul Samuelson (1989, p. 100) declared with reference to the 1937 paper, explaining further on that the genius of von Neumann's contribution fitted *any* capital model. von Neumann (1945/46, p. 2) himself cleared the question of the filiations in a lapidary (and, after the fact, ironic) style: "It is obvious to what kind of theoretical models the above assumptions correspond", as if this was not the issue at stake, drawing attention once more to the technical aspects and the nature of the mathematical approach itself.

2. Von Neumann and the Formalist Programme of Hilbert: Before and After Gödel

From the start, a significant problem seems to threaten our interpretation. It is of chronological order. The article of 1937 was designed, then published after von

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[3] To support this assertion, Chakravarty indirectly leaned on the book review Morgenstern wrote in 1941 on *Value and Capital* by Hicks. In a biographical note, in fact, it appears that Morgenstern submitted his review to the previous reading of von Neumann. It is possible to read here that the main criticism addressed to Hicks regards precisely the kind of mathematical techniques used to prove the existence of an economic equilibrium. Cf. Robert Leonard (1995) for a detailed analysis of the collaboration between Morgenstern and von Neumann, and, more precisely, for an analysis of the extent of the intellectual influence of von Neumann upon Morgenstern.

Neumann was informed of the famous theorem of impossibility of Gödel, devastator of the mathematical formalist programme and unanimously recognised as an element of rupture in the evolution of modern mathematics. von Neumann is also one of the first mathematicians to seize the range of Gödel's theorem and to take into consideration its methodological consequences. It is necessary at this level to reconsider the definition of the formalist Hilbertian programme in order to understand more precisely what the impact of Gödel's discoveries was, and to what extent it modified mathematical practices.[4]

The term "formalism" itself is ambiguous because it bears a double significance. In its commonly accepted sense, formalism indicates nothing other than the mere use of symbols and unspecified mathematical techniques to express an idea. It is not acceptance that this term implies when it is associated with Hilbert. By formalism, one then understands a particular philosophy of mathematics which reduces it to a formal language, and is opposed to intuitionism and logicism on the question of the foundations of mathematics.

The debate on the foundations emerges among mathematicians at the end of the nineteenth century, while attempts to extend the traditional axiomatic (Euclidean) method to branches of mathematics other than geometry are multiplying. This method consists in accepting without demonstration a reduced set of postulates, the axioms, and deducing by logical inference a set of theorems. For a long time, the empirical obviousness of axioms seemed to guarantee the veracity of the theorems which it was possible to deduce. But the growing abstraction of the mathematical practice (axioms are less and less obvious) and the discovery by Cantor and Russell of logical antinomies (even if axioms were obvious, contradictions could emerge) bring to the foreground the question of the consistency of formal systems. "Consistency" refers to a precise property: a formal system is consistent when it is impossible to deduce from its axioms two contradictory theorems. Three types of answer were advanced to give back to mathematicians their confidence in the rigour of mathematical practices.

Logicists try to found the consistency of mathematics by defining it as a branch of logic. The *Principia Mathematica* of Whitehead and Russell, published in 1910, falls under this head. There, the authors proposed a formalisation of arithmetic, whose goal is to clarify and make explicit all the logical inferences used in the reasoning and to show that all the concepts of arithmetic can be brought back to concepts of pure logic. However, this step did not gain much support from mathematicians as this solution did nothing but move the problem: the consistency of arithmetic depended on that of logic, and the consistency of logic was then itself under discussion.

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[4] Gödel's theorems certainly changed von Neumann's views drastically about what constituted the standard of good practices in mathematics, but such changes had not been the only ones the mathematician experienced during his career. For instance, as suggested by Giocoli (2003), a crucial issue is to understand why, in his 1944 book, von Neumann provided Morgenstern constructive proof of the Minimax Theorem and turned to defend classical axiomatics. This interesting question is, however, beyond the realm of this paper.

Intuitionists, headed by Poincaré and Broüwer, placed the authority of the perception and of the intuition of the mathematician above that of the logical principles and inference rules whose historical and cultural relativity were underlined. To be consistent, a system of calculation must thus be built from obvious and unimpeachable axioms and from rules of inference subjectively considered as reliable by the mathematician.[5] To Luitzen Broüwer (1912, p. 125), the fundamental dissension which exists between intuitionalism and formalism is that a different answer is given to the question of knowing where the mathematical accuracy exists: to the intuitionalist, in human intellect; to the formalist, on paper. Thus, the consistency of a mathematical theory does not require a demonstration for intuitionalists insofar as it results from the construction itself of the theory, following the principles and the procedures acceptable to the majority of mathematicians.

On the contrary, the response of formalists to the uncertainty on foundations consisted of trying to establish rigorous evidence of consistency of the various branches of mathematics. Demonstrations of consistency initially take the form of relative proofs. Thus, Hilbert showed that the consistency of Euclidean geometry depends on that of algebra. Thereafter, he tried, with the assistance of his disciples (the first of whom was von Neumann, but also Ackermann and Bernays) to provide an absolute demonstration of consistency of arithmetic.[6] It is at this level that the famous impossibility theorem of Gödel intervenes. In 1931, Gödel arrived at a devastating result on the question of the foundations of mathematics. He, in fact, showed that it was impossible to provide a demonstration of absolute consistency of arithmetic.[7] Gödel did not prove the inconsistency of arithmetic, rather, the impossibility of showing that it was consistent, leaving the door open to the potential occurrence of new logical antinomies. In his book of reference on the question, Morris Kline (1980) presented in a provocative way the debate on the foundations of mathematics as a major intellectual rout, liquidating the hitherto-dominant design of mathematics like point of organ of rigour and scientific exactitude. The title of his work, *The Loss of Certainty*, returned precisely to this radical reconsideration: mathematics cannot be unanimously regarded any more as a set of firmly established eternal truths.

This result certainly cooled down the enthusiasm of formalists but did not put an end to the programme of Hilbert whereof the work on foundations constitutes only one part. Formalists gave up the hope to be able to show that mathematics were con-

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[5] As a matter of fact, intuitionalists reject the logical principle of exclusion of the middle for infinite systems. On this subject, Hilbert wrote in 1928 in his *Die Grundlagen der Mathematik* that to remove this principle from the toolbox of mathematicians would be the same thing as prohibiting use of the telescope for astronomers, or boxers the use of their fists.

[6] This problem appears in number 2 on the famous list of the 23 problems of Hilbert. At the time of the Second International Congress of Mathematics held in Paris in 1900, Hilbert enumerated a list of 23 major, irresolute mathematical problems, which, according to him, would be solved during the twentieth century to allow important progress in the discipline. The optimism of Hilbert will never fade. At the time of a noted radio intervention in 1930, he launched his famous "Wir müssen wissen, wir werden wissen" ("We must know, we will know"). By today, most of these problems have been solved, and Hilbert's list has been replaced by the seven Millennium Prize problems.

[7] Gödel actually led to two results in his 1931 article: (1) for any formal system including arithmetic, it is possible to construct a proposition which is true in this system, but not provable in it; and (2) one cannot prove the consistency of a formal system containing a finite theory of numbers.

sistent, but they did not give up their confidence in the power of modern axiomatics as an engine for discovering new scientific knowledge. As Giorgio Israel and Ana Gasca (1995) note indeed, the formalism of Hilbert was founded on the belief in a pre-established harmony between mathematics and physical reality, a harmony which makes it possible to conceive mathematics like the base of all exact scientific knowledge of nature. The normative aspect of Hilbert's programme can consequently be interpreted as follows: the mathematical analogy, understood as the systematic adoption of the modern axiomatic approach represents the good scientific practice and this, whatever the scientific field considered.

> *I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms . . . we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science.*

<div align="right">(Speech by Hilbert 1918, in William B. Ewald 1996; and Roy E. Weintraub 1998)</div>

The association between axiomatic method and scientific rigour thus justifies the second side of the formalist programme of Hilbert consisting concretely of trying to extend this approach to other scientific disciplines, physics initially, but also economics.[8] Thus, Hilbert's formalism has a double finality: to solve the problem of the foundations of mathematics (and, at this level, the results of Gödel are without call); and to extend modern axiomatics to all the scientific disciplines. This second aspect of the programme, the aspect that can be described as the imperialist or normative side, survived to Gödel. Weintraub (2002, p. 90) identified these two aspects of the formalist programme.[9] He distinguished between the Finitist Programme for the Foundations of Arithmetic (FPFA) whose objective was to found the consistency of arithmetic and the AA (axiomatic approach), the only aspect of the formalist programme which has actually influenced the process of mathematisation of economics through the contributions of von Neumann for the strictly Hilbertian version of the AA programme, and Debreu for the Bourbakist version.[10]

Until 1931, von Neumann was strongly implicated in the two aspects of Hilbert's formalist programme. As far as the work on foundations is concerned, he contributed to the axiomatisation of Cantor's set theory. This theory, known as the "naïve" theory of sets because it was then not yet in axiomatic form, leads to logical inconsistencies discovered around 1900 by Cantor himself and by Russell. Since his doctorate thesis, von Neumann contributed to looking further into the axiomatisation of set theory proposed by Zermelo, Fraenkel and Skolem through the introduction of new axioms and methods, making it possible to avoid the occurrence of these contra-

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[8] Let us note, besides, that if the first two points of Hilbert's list of 1900 relate to the question of the foundations of mathematics, item 6 invites to axiomatisation of physics on the model of mathematics.

[9] Leonard (1995, p. 732) also puts ahead these two aspects of the formalist programme of Hilbert: to base all the branches of mathematics on a sure axiomatic base; and to extend axiomatics to other fields.

[10] For an account of the differences between the Hilbertian and Bourbakist versions of formalism applied to economics, see Philippe Mongin (2003).

dictions. The axiomatic method is used in order to allow a rigorous representation of the theory within which the origin of contradictions can be easily found and possibly eliminated.

Regarding the normative aspect of the formalist programme, since 1926 von Neumann tackled the question of the mathematical axiomatisation of quantum physics, then defined around the two competing presentations of Heisenberg and Schrödinger. This work led to the publication in 1932 of the *Mathematical Foundations of Quantum Mechanics* in which the author managed to unify these two visions within a single formal system. Game theory is another field where the project of exporting modern axiomatics to new fields of scientific knowledge appears: von Neumann followed at the beginning the developments of Zermelo on the axiomatisation of chess, a question much debated in discussions in mathematical circles of the inter-War period. It was a question of showing that a formal system could receive an interpretation in terms of social phenomena rather than in strictly natural terms. von Neumann generalised the application of Zermelo to the context of any type of zero-sum games, and this work led him to the determination of the Minimax Theorem in 1928. From there on Hilbertian formalism could penetrate the field of individual interactions and be used for the analysis of social phenomena.

3. The Pragmatic Turn

Gödel's discoveries affected von Neumann deeply. They contributed to immediately putting a term to his work on the foundations of mathematics and signalled the beginning of what many commentators describe as a pragmatic turn in the scientist's method.[11] Hilbert's programme on the foundations conveyed the hope of justifying the axiomatic method, to carry mathematical results to the statute of eternal truth. Gödel destroyed this hope, but the majority of mathematicians (von Neumann among them) decided to use this method all the same because it remained, in spite of the loss of certainty, a rigorous way of producing scientific knowledge. The second side of Hilbert's programme was unharmed.

> *The main hope of a justification of classical mathematics – in the sense of Hilbert or of Brouwer and Weyl – being gone [Gödel's discoveries], most mathematicians decided to use that system anyway. After all, classical mathematics was producing results which were both elegant and useful, and, even though one could never again be absolutely certain of its reliability, it stood on at least as sound a foundation as, for example, the existence of the electron. Hence, if one was willing to accept the sciences, one might as well accept the classical system of mathematics. Such views turned out to be acceptable even to some of the original protagonists of the intuitionistic system. At present the controversy about the "foundations" is certainly not closed, but it seems most unlikely that the classical system should be abandoned by any but a small minority.*

(von Neumann 1947, p.194)

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[11] Cf., for instance, Israel and Gasca (1995, p. 63).

That said, if, after Gödel, it was accepted that it was impossible to found mathematics absolutely, however, indirect ways existed to comfort scientists and to relativise the loss of certainty they suffered in full measure. First of all, should a contradiction emerge, formalisation makes it easier to search for its origins and eventually to eliminate it thanks to the baring of all of the concepts and reasoning intervening in the theory. The position of the Bourbakist programme is for this reason evocative: the objective of this radical version of formalism is not to found mathematics any more, rather, to clarify, through the linking of formal systems with one another, the architecture and unity of mathematics. The mathematician must face contradictions, if they emerge, on a case-by-case basis.

> *Absence of contradiction, in mathematics as a whole or in any given branch of it, thus appears as an empirical fact, rather than as a metaphysical principle. The more a given branch has been developed, the less likely it becomes that contradictions may be met with in its further development. [...] What will be the working mathematician's attitude when confronted with such dilemmas? It need not, I believe, be other than strictly empirical. We cannot hope to prove that every definition, every symbol, every abbreviation that we introduce is free from potential ambiguities, that it does not bring about the possibility of a contradiction that might not otherwise have been present. Let the rules be so formulated, the definitions so laid out, that every contradiction may most easily be traced back to its cause, and the latter either removed or so surrounded by warning signs as to prevent serious trouble. This, to the mathematician, ought to be sufficient;*

(Nicolas Bourbaki, 1949, p.3)

There is a second means of reassuring the scientist about the consistency of his formal system. It consists of putting back to the foreground considerations of a semantic nature. This assertion requires further elaboration. A prominent characteristic of Hilbertian formalism is without any doubt the strict separation between syntax and semantics. To formalise a theory in the sense of Hilbert means indeed emptying it from all of its semantic content and giving an abstract representation of it – the formal system – in the form of symbols, formulae (among them axioms) and sequences of formulae having no more obvious bond with the theory of departure. The formal system thus formed is like an abstract box, deprived of any significance, on which the mathematician works in order to draw theorems. At this stage, the question of the realism of the axioms is completely irrelevant. But it would be erroneous to say that in axiomatics reality does not matter at all, for in the next stage of the axiomatisation process, the objective is precisely to assign models to each formal system, that is, to find an interpretation in terms of real phenomena for the formal system.[12] A model consists of an interpretation of the formal system, each symbol receiving a meaning, and the same abstract box being able to receive various interpretations. The initial theory which inspired the formal system constitutes one model, among others. Formalism as a philosophy of mathematics is attached at this level with Plato's realism consisting of supporting the thesis that mathematics does not create anything, does not invent objects, rather, discovers pre-existent objects in the intellect. The power of

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[12] Leo Corry (2004) insists on the concern Hilbert presents for the realism of his axiomatic approach.

axiomatisation is due precisely to the fact that the "discovery" of an abstract box makes it possible to explain several real phenomena, and rests on the belief of a pre-set adequacy between the structure of mathematics and reality.

> *From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms – the mathematical structures; and so it happens without our knowing how that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation.*
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> (Bourbaki 1950, p.231)

This vision of the world is opposed to constructivism, of which intuitionism is a specific form, and which considers that a mathematical object exists only through its elaboration. To formalists, on the contrary, the very existence of any mathematical concept refers to a precise property: that it is free from any contradiction.

Before paradoxes and logical antinomies were discovered and encouraged mathematicians to work out absolute demonstrations of consistency, it was sufficient, in order to found a formal system, to find a model in which its axioms were valid. For a long time, the obviousness of the Euclidian axioms was sufficient to ensure the consistency of Euclidian geometry: if axioms were valid, then it was also the case for the theorems that one could derive from them. The so-called *method of the models* consisting of finding an interpretation to an abstract system in which its postulates are valid was largely used to give relative demonstrations of consistency to formal systems less intuitive than the Euclidean one. Gödel's discoveries led mathematicians to reconsider the value of this method. One cannot found the consistency of a formal system absolutely, but the discovery of a new and adequate model for this system reinforces its heuristic validity and comforts the mathematician regarding its consistency. The 1937 contribution of von Neumann may be interpreted in that way: *a new semantic correspondence is associated with a formal system elaborated beforehand.* In particular, von Neumann gave an economic interpretation to a formal structure which he previously discovered in game theory (1928). This idea was expressed explicitly by the author himself when he declared that "the question whether our problem has a solution is oddly connected with that of a problem occurring in the Theory of Games dealt with elsewhere" (von Neumann, 1945/46, p. 33, n. 1).

The formal similitude between the 1928 and 1937 models is, however, not immediate. In 1928, von Neumann demonstrated the existence of a solution for a two-person zero-sum game without ever defining a system of linear inequalities and equations. As Tinne H. Kjeldsen (2001) states, the Minimax Theorem was developed in 1928 with no explicit connection with the theory of linear inequalities, and there are no elements that show that von Neumann would be aware at that time of this connection. However, the fact that this connection does exist is sufficient to corroborate this rational reconstruction. Kuhn and Tucker (1958) explicitly link the solutions of the minimax problem with a system of linear inequalities and equations which corresponds to the problem raised in 1937. They state explicitly that if the intensity and price vectors are both normalised, they form probability vectors which may be regarded as mixed strategies for the players of a zero-sum two-person game. Dore (1989b) also studied the connection between the system of inequalities and equations

of the 1937 model and the two-person zero-sum game of 1928: the strategies of player I are represented by the set of vectors of production intensities, those of player II by the set of price vectors. Payoff functions depend on the strategies chosen by each player: player I chooses the vector of the intensities of production which maximises his payoff function, given the choice of player II, supposed for his part to choose the least satisfactory solution for the first player. A symmetrical reasoning relates to the choices of player II. The Minimax Theorem ensures the existence of a saddle point which corresponds to the situation where the rate of growth is equal to the interest rate.

The 1937 article illustrates the separation and hierarchy between syntax and semantics, typical of the axiomatic approach. The same formal system, the same box, indeed receives different interpretations, i.e. different models: one in game theory, one in economics, and even one in thermodynamics.[13] Thanks to Gödel, we know that the consistency of this formal system is impossible to prove. However, the fact that this system fits different interpretations is a reassuring symptom of its consistency. The economic interpretation is, in this connection, the manifestation of the pragmatic turn of the mathematical formalist programme which consisted in considering not only the syntax aspect, but also the semantic step of the axiomatisation process through the identification of adequate new models. Further, with the 1937 paper, a new domain of application, economics, opened itself up to formalist mathematics, and, more generally, to mathematical analogy.

4. From the Mechanical to the Mathematical Analogy

The growth model was elaborated in 1931 in the United States and first presented to a mathematical seminar at Princeton, but it has definitely been arousing interest and enthusiasm since its discussion in the Karl Menger seminar in Vienna in 1934. One reason for the particular interest of Viennese scholars in the growth model lies in the total adequacy between von Neumann's epistemological approach in this paper and the specific philosophical context of the Vienna Circle, marked by analytical philosophy, logical positivism, and a project of unification of sciences.

One finds a definite parallelism between the concerns of formalist mathematicians on one side and of logical positivist philosophers on the other. The major concern of mathematicians is to eliminate the possibility of contradictory theorems; the major concern of philosophers is to eliminate from their discourse all metaphysical proposition, i.e. any pseudo-scientific assertion whose intrusion in the reasoning may lead to logical inconsistencies. In both cases, discussions are directed towards the research of certainty in scientific reasoning.

The principal theses of logical positivism are presented by Otto Neurath, Rudolf Carnap, and Moritz Hahn in an article of 1929, "The Scientific Conception of the World: The Vienna Circle", better known as the "Manifesto of 29". Logical posi-

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[13] The thermodynamic interpretation is mentioned by von Neumann only (1945/46, p. 29), who suggests an interpretation of function $\Phi(X, Y)$ in terms of thermodynamic potentials. This way is explored by Brody (1989), who considers economic processes as chemical processes, whose speeds of reaction correspond to the intensities of production.

tivism falls under the continuation of the positivist programme of Auguste Comte, Hume and Mach, whose objective was to base knowledge directly on experience. To this end, members of the Vienna Circle used the latest developments of modern logic from Frege, Peano and Russell. More precisely, logical positivism was born from the introduction of logical analysis into the positivist framework. Logical analysis consists in reducing scientific concepts and propositions to experience, to direct observation, from which all the remainder logically arises. In the same way that axiomatisation makes it possible to uncover the source of possible contradictions easily, logical analysis tracks pseudo-propositions and contributes to eliminating them from philosophical discourse. The project of Carnap is even more ambitious. The philosopher has been working on a project to work out a formal logico-mathematical language used to guard scientists against the surreptitious intervention of pseudo-propositions in their reasoning. Philosophy thus becomes analytical: it is finalised with the revelation of the significance of propositions and the elimination of meaningless propositions. This "turning point of philosophy" (Schlick 1959, p. 56) is an indicator of the ambition of logical positivism to aim at unitary science. With analytical philosophy, it will not be necessary any more to speak about philosophical problems, because all problems will be discussed philosophically, i.e. clearly and meaningfully. The call for the unity of science, explicit in the Manifesto, claims to be epistemological. It is a means for scientists of working out a way of making science, whatever be the field of production of knowledge, which ensures rigorous reasoning, free from metaphysics. This is logical analysis for Russell, the universal formal language for Carnap, and modern axiomatics for Hilbert.

The unifying ambition of formalism asserts itself gradually. Initially, it was a question of unifying, through the development of modern axiomatics, all the branches of mathematics. Formalists, rather, their predecessors, analyticals, were then opposed to the purist vision of mathematics dominant by the end of the nineteenth century. According to purists, mathematics was to remain split in various branches, each defined by its own method of investigation. For example, purists refused geometric demonstrations based on Cartesian algebra. Analyticals, on the contrary (with Hilbert in the forefront), believed in the interaction of the various branches and shared an ideal of unification of mathematics, conceived as a unified system of knowledge. In a second step, this strong optimism exceeded the borders of the discipline; building from the success of the axiomatisation of quantum physics, formalists then invested the field of social phenomena.

Economics is implied in the philosophical programme of the Vienna Circle through the active interaction of the members of Hans Mayer's Economic Seminar with those of the Mathematical Colloquium run by Menger, son of the founder of the Austrian economic tradition. Collaboration between mathematicians and economists crystallised in the resolution of the problem of imputation as defined by Menger in 1871. It consists of deducing the prices of factors of production starting from the value of the consumption goods which they contributed to produce. The solution suggested in 1889 by Wieser encounters a problem of surdetermination. Schlesinger, asked by Mayer to harness himself with the question, radically modified the nature of the problem: he endogenised the prices of consumption goods that Menger and Wie-

ser took as data and posed the equations of a system of generalised interdependence. The question of imputation thus becomes that of demonstrating the existence of a general equilibrium configuration. Schlesinger, however, did not start from the Walrassian model, but from the very similar one of Gustav Cassel (1923), in which he integrated the free goods rule in order to avoid obtaining negative prices in equilibrium. The adoption of this rule has important consequences on the formal structure of the model: inequalities are introduced into the model; inequalities are relations of exclusion which constrain the prices of goods and which have the statute of axioms in the formulations offered by the mathematicians (Abraham Wald and later von Neumann) called to the rescue to solve the new system thus defined. The introduction of inequations is typical of formalist mathematics. According to Israel and Gasca (1995, p. 65), the motto "less differential equations, more inequalities" perfectly describes the tendency of the new mathematics.[14]

From his collaboration with Schlesinger, Wald produced three articles, presented at the Mathematical Colloquium between 1934 and 1936.[15] Over the course of the various articles, the mathematician refined the mathematical conditions necessary for the demonstration of existence (the syntax aspect) and concentrated himself more particularly on the question of their economic significance (the semantic aspect).[16] von Neumann became aware of Wald's demonstrations thanks to Menger in 1934 and announced the proximity with a model of general equilibrium which he had presented a few times earlier at Princeton. Menger then made an offer to von Neumann to publish his article in *Ergebnisse* (1937). According to Arrow (1989, p. 17), it is extremely probable that the models of Schlesinger and Wald on one hand and of von Neumann on the other were independently inspired by Cassel. Whereas Schlesinger introduced inequalities in the static model of Cassel, and Wald showed the existence of an equilibrium solution, von Neumann's model axiomatises the verbal developments Cassel made of an economy of generalised interdependence in a situation of uniform growth. Nicholas Kaldor (1989, p. viii) said, from his conversations with von Neumann, that the dissatisfaction of the mathematician with regard to the Walrassian model had a double origin: the possibility of negative prices at equilibrium, and the disinterest in dynamic forces. The 1937 model answered these two criticisms appropriately by proposing a model of expansion in which the free goods rule, with the statute of axiom of the formal system, eliminated the possibility of negative prices in equilibrium.

The 1937 model, however, also addressed a more general criticism to economists.[17]

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[14] In a letter to Arrow (1989, p. 25), Leijonhufvud relates the following anecdote on this subject. At the time of a multidisciplinary seminar in 1930 in Berlin on the application of mathematics to various fields, von Neumann had stopped the presentation of a general equilibrium model by Marshak to vigorously criticise the use of equations: "but surely you want inequalities, not equations there!" This anecdote is also related by Weintraub (1983, p. 13).

[15] A last paper from Wald would have been lost (cf. Arrow, 1989).

[16] For an evaluation of the work of Wald in terms of contribution to the Viennese philosophical project, see Philippe Le Gall (1991).

[17] After the presentation of the growth model at the Princeton Mathematical Society in 1932, Abraham Flexner, then director of the Princeton Institute for Advanced Study, sent von Neumann a copy of a book

I have the impression that [economics] is not yet ripe (I mean is not yet fully enough understood, which of its features are the essential ones) to be reduced to a small number of fundamental postulates – like geometry, or mechanics…

(von Neumann to Abraham Flexner, May 25, 1934,
Faculty files, John von Neumann, folder 1933-35, VNIAS)

According to Leonard (1995, p. 738), the fundamental criticism of von Neumann here related to the kind of mathematical instruments used since Walras in economic formalisation. However, if one replaced the 1937 contribution within the second part of the formalist programme of Hilbert (the imperialist aspect of the programme, with its project of extension of modern axiomatics to various fields), then, more than the type of tools used, it is the concept itself of scientific rigour which seems to be at the heart of von Neumann's criticisms on the state of the discipline. Walras used the mechanical analogy with the stated aim of giving economics the scientific rigour which was lacking till that point. Walrassian economics, like the other sciences based on the mechanical analogy, adopts as scientific criterion of rigour confrontation with reality. Accordingly, a model is an economy in miniature which is sufficiently simplified to allow mathematical treatment. The adoption of the mathematical analogy radically modifies this perception. Scientific rigour is defined according to internal criteria, mainly aesthetical (von Neumann 1947); rigour becomes synonymous with purity, abstraction, and consistency of the formal system. Certainly, scientific rigour is a relative and changing concept. Thanks to Gödel, von Neumann paid the price.[18] In the ultimate analysis, Gödel's discoveries resounds like a bulwark against possible drift towards abstraction, of which Hilbertian formalism could be the thin end of the wedge.

As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality" it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l'art pour l'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then

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of George and Edouard Guillaume, *L'Économique Rationnelle*, at the origin of these criticisms on the state of economics. The authors gave a mathematical representation of a production economy explicitly formalised on the basis of a strict analogy with physics. This episode is reported in detail by Leonard (1995, p. 736).

[18] "I have told the story of this controversy [on the foundations of mathematics] in such detail, because I think that it constitutes the best caution against taking the immovable rigor of mathematics too much for granted. This happened in our own lifetime, and I know myself how humiliatingly easily my own views regarding the absolute mathematical truth changed during this episode, and how they changed three times in succession!" (von Neumann 1947, p.195).

the danger signal is up. It would be easy to give examples, to trace specific evolutions into the baroque and the very high baroque, but this, again, would be too technical.

(von Neumann 1947, p.195)

Of course, these critics are not concerned specifically with economics but with the most abstract practices of mathematicians, as, for instance, in the Bourbakist programme, the radical extension of formalism. But by substituting the term "mathematical" by "economical" in the preceding quotation, the criticism remains valid to some extent, testifying to the success of the imperialist incursion of formalism in economics.

The thesis of this paper is that the 1937 article is a contribution to the mathematical formalist programme. We defined this programme around two finalities: the search for certainty, and the project of unifying sciences. After Gödel's discoveries, the first part of the programme has faded deeply, whereas the second aspect remains intact. At the end of our reflection, it seems to us that the 1937 article fully fits the second aspect of this programme and reflects to a certain extent its new pragmatic dimension. We indeed tried to show that, *a posteriori*, von Neumann's 1937 contribution fulfils a twofold motivation:

- To find a new model of a formal system insofar as, if it is not possible to prove the consistency of a system, it is nevertheless possible to consolidate the certainty of scientists through the exhibition of a new adequate interpretation;
- To replace the use in economics of the mechanical analogy by the mathematical analogy.

Admittedly, much has already been written on the "most important paper done in mathematical economics" (Weintraub 1985, p. 27; and 2002, p. 95). It was disguised with the most various interpretations. Ours is a contribution to the more restricted set of comments which concentrate less on possible filiations of the model than on the range of the original methodological approach of the author, positioning the 1937 contribution in the formalist revolution in economics. On this subject, von Neumann was, in those days, an enlightened defender of modern axiomatics, conscious of the possible drifts of formalist practices towards "the baroque", towards "*l'art pour l'art*", and it seems that his warnings concern economists very much today.

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