A FIXED POINT THEOREM FOR MULTI-MAPS SATISFYING AN IMPLICIT RELATION ON METRIC SPACES

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We present a fixed point theorem for multi-valued mapping satisfying an implicit relation on metric spaces.

1. INTRODUCTION AND PRELIMINARIES

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach’s fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach’s fixed point theorem in different ways. In [6], Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (see, [3], [5], [7]–[13].

Throughout this paper, let \((X, d)\) be a metric space. Also \(B(X)\) is the set of all non-empty bounded subsets of \(X\). Denote for \(A, B \in B(X)\)

\[
D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},
\]

\[
\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.
\]

If \(A\) consists of a single point \(a\), we write \(\delta(A, B) = \delta(a, B)\). If \(B\) also consists of a single point \(b\), we write \(\delta(A, B) = d(a, b)\).

\[
H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}
\]

2000 Mathematics Subject Classification. 54H25, 47H10.

Keywords and Phrases. Fixed point, weakly compatible maps, multi-maps, implicit relation.
for $A, B \in CB(X)$, where $CB(X)$ is the set of all non-empty closed and bounded subsets of $X$. Note that $D(A, B) \leq H(A, B) \leq \delta(A, B)$. Function $H$ is a metric on $CB(X)$ and is called a Hausdorff metric. It is well known that if $X$ is a complete metric space, then so is the metric space $(CB(X), H)$. The following definition is given by Jungck and Rhoades [7].

**Definition 1.** The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e., for each point $u$ in $X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$. (Note that the equation $Fu = \{Iu\}$ implies that $Fu$ is a singleton).

2. **IMPLICIT RELATION**

Implicit relation on metric space have been used in many articles (see [1], [2], [4], [9], [13]).

**Definition 2.** Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ be the set of all non-negative real numbers and let $T$ be the set of all continuous functions $T : (\mathbb{R}^+)^5 \to \mathbb{R}$ satisfying the following conditions:

1. $T(t_1, \ldots, t_5)$ is non-decreasing in $t_1$ and non-increasing in $t_2, \ldots, t_5$.
2. There exists $h \in (0, 1)$ such that
   \[
   T(u, v, u, v, u + v) \leq 0 \quad \text{or} \quad T(u, v, u, v, v + u) \leq 0
   \]
   implies $u \leq hv$.
3. $T(u, 0, 0, u, u) > 0, T(u, 0, 0, u, u) > 0$ and $T(u, 0, 0, 2u) > 0$, for all $u > 0$.

Now, we give some examples.

**Example 1.** Let $T(t_1, \ldots, t_5) = t_1 - \alpha \max\{t_2, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$.

1. **(C1):** Obvious. **(C2):** Let $u > 0$ and $T(u, v, u, u, v + u) = u - \alpha \max\{u, v\} - \beta(v + u) \leq 0$. Then $u \leq \alpha + \beta u$, a contradiction. Thus $u < v$ and $u \leq (\alpha + \beta)v + \beta u$ and so $u \leq \frac{\alpha + \beta}{1 - \beta} v$. Similarly, let $u > 0$ and $T(u, v, u, v, v + u) = u - \alpha \max\{u, v\} - \beta(v + u) \leq 0$, then we have $u \leq \frac{\alpha + \beta}{1 - \beta} v$.

If $u = 0$, then $u \leq \frac{\alpha + \beta}{1 - \beta} v$. Thus **(C2) is satisfying with $h = \frac{\alpha + \beta}{1 - \beta} < 1$.** **(C3):** $T(u, 0, 0, u, u) = T(u, 0, 0, u, u) = u(1 - \alpha - \beta) > 0$ and $T(u, 0, 0, 2u) = u(1 - \alpha - 2\beta) > 0$, for all $u > 0$. Therefore $T \in T$.

**Example 2.** Let $T(t_1, \ldots, t_5) = t_1 - m \max\{t_2, t_3, t_4, t_5/2\}$, where $0 \leq m < 1$.

1. **(C1):** Obvious. **(C2):** Let $u > 0$ and $T(u, v, v, u, v + u) = u - m \max\{u, v\} \leq 0$. Thus $u \leq m \max\{u, v\}$. Now if $u \geq v$, then $u \leq mu$, a contradiction. Thus $u < v$ and $u \leq mv$. Similarly, let $u > 0$ and $T(u, v, u, v, v + u) = u - m \max\{u, v\} \leq 0$, then we have $u \leq mv$. If $u = 0$, then $u \leq mv$. Thus **(C2) is satisfying with $h = m < 1$.** **(C3):** $T(u, 0, 0, u, u) = T(u, 0, 0, u, u) = T(u, u, 0, 0, 2u) = u(1 - m) > 0$, for all $u > 0$. Therefore $T \in T$. 
3. THE MAIN RESULT

Theorem 1. Let $F,G$ be mappings of a complete metric space $(X,d)$ into $B(X)$ and $f,g$ be mappings of $X$ into itself satisfying:

(i) $Fx \subseteq g(X)$, $Gx \subseteq f(X)$ for every $x \in X$,
(ii) The pair $(F,f)$ and $(G,g)$ are weakly compatible,
(iii) $T(\delta(Fx,Gy),d(fx,gy),D(fx,Fx),D(0,Gy),D(fx,Gy)+D(0,Fx)) \leq 0$

for every $x,y \in X$, where $T \in T$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of $X$, then there exists a unique $p \in X$ such that $\{p\} = \{fp\} = \{gp\} = Fp = Gp$.

Proof. Let $x_0$ be an arbitrary point in $X$. By (i), we choose a point $x_1$ in $X$ such that $y_0 = gx_1 \in Fx_0$. For this point $x_1$ there exists a point $x_2$ in $X$ such that $y_1 = fx_2 \in Gx_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$y_{2n} = gx_{2n+1} \in Fx_{2n}, \quad y_{2n+1} = fx_{2n+2} \in Gx_{2n+1},$$

for $n = 0, 1, 2, \ldots$. We prove that sequence $\{y_n\}$ is a CAUCHY sequence. From (iii), we have

$$T(\delta(Fx_{2n},Gx_{2n+1}),d(fx_{2n},gx_{2n+1}),D(fx_{2n},Fx_{2n}),D(gx_{2n+1},Gx_{2n+1}),\quad D(fx_{2n},Gx_{2n+1})+D(gx_{2n+1},Fx_{2n})) \leq 0.$$

Using (C$_1$) we get

$$T(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\quad d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})) \leq 0$$

and so we get

$$T(d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),d(y_{2n-1},y_{2n}),d(y_{2n},y_{2n+1}),\quad d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})) \leq 0,$$

that is

$$T(u,v,u,v+u) \leq 0,$$

where $u = d(y_{2n},y_{2n+1})$ and $v = d(y_{2n-1},y_{2n})$. Hence, from (C$_2$), there exists $h \in (0,1)$ such that

$$d(y_{2n},y_{2n+1}) \leq hd(y_{2n-1},y_{2n}).$$

Similarly, from (iii), we have

$$T(\delta(Fx_{2n+2},Gx_{2n+1}),d(fx_{2n+2},gx_{2n+1}),D(fx_{2n+2},Fx_{2n+2}),D(gx_{2n+1},Gx_{2n+1}),\quad D(fx_{2n+2},Gx_{2n+1})+D(gx_{2n+1},Fx_{2n+2})) \leq 0.$$
Thus we have
\[
T \left( d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2}) \right) \leq 0.
\]

Using (C₁) we have
\[
T \left( d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2}) \right) \leq 0,
\]
That is
\[
T(u, v, u, v, v + u) \leq 0,
\]
where \( u = d(y_{2n+2}, y_{2n+1}) \) and \( v = d(y_{2n+1}, y_{2n}) \). Hence, from (C₂), we have
\[
d(y_{2n+2}, y_{2n+1}) \leq hd(y_{2n+1}, y_{2n}).
\]
Therefore,
\[
d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq \cdots \leq h^n d(y_0, y_1),
\]
Thus
\[
d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m)
\]
\[
\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + \cdots + h^m d(y_0, y_1)
\]
\[
= \frac{h^n - h^m}{1 - h} d(y_0, y_1)
\]
\[
\leq \frac{h^n}{1 - h} d(y_0, y_1) \to 0.
\]

Hence the sequence \( \{y_n\} \) is a Cauchy sequence in \( X \). By completeness \( X \) there exist \( p \in X \) such that
\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{2n} = \lim_{n \to \infty} g(x_{2n+1}) = p \in \lim_{n \to \infty} Fx_{2n},
\]
and
\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} f(x_{2n+2}) = p \in \lim_{n \to \infty} Gx_{2n+1}.
\]

Suppose that \( g(X) \) is closed, then for some \( v \in X \) we have \( p = gv \in g(X) \). If set \( x_{2n}, v \) replacing \( x, y \) respectively, in inequality (iii) we get
\[
T \left( \delta(Fx_{2n}, Gv), d(fx_{2n}, gv), D(fx_{2n}, Fx_{2n}), D(gv, Gv), D(fx_{2n}, Gv) + D(gv, Fx_{2n}) \right) \leq 0.
\]
From \((C_1)\), we have

\[
T \left( \delta(y_{2n}, Gv), d(y_{2n-1}, gv), d(y_{2n-1}, y_{2n}), D(gv, Gv),
D(y_{2n-1}, Gv) + d(gv, y_{2n}) \right) \leq 0.
\]

Letting \(n \to \infty\), we have

\[
T \left( \delta(p, Gv), d(p, gv), d(p, p), D(p, Gv), D(p, Gv) + d(p, p) \right) \leq 0.
\]

Thus from \(C_1\) we get,

\[
T \left( \delta(p, Gv), 0, 0, \delta(p, Gv) \right) \leq 0.
\]

That is, \(T(u, 0, 0, u, u) \leq 0\), hence from \((C_3)\), we get \(u = \delta(p, Gv) = 0\). Hence \(Gv = \{p\} = \{gv\}\). From weak compatibility of \((G, g)\), we have \(Ggv = gGv\), hence \(Gp = \{gp\}\). If set \(x_{2n}, p\) replacing \(x, y\) respectively, in inequality (iii) we get

\[
T \left( \delta(Fx_{2n}, Gp), d(fx_{2n}, gp), D(fx_{2n}, Fx_{2n}), D(gp, Gp),
D(fx_{2n}, Gp) + D(gp, Fx_{2n}) \right) \leq 0.
\]

From \((C_1)\), we have

\[
T \left( d(y_{2n}, gp), d(y_{2n-1}, gp), d(y_{2n-1}, y_{2n}), d(gp, gp), d(y_{2n-1}, gp) + d(gp, y_{2n}) \right) \leq 0.
\]

Letting \(n \to \infty\), we get

\[
T \left( d(p, gp), d(p, gp), d(p, p), d(gp, gp), d(p, gp) + d(gp, p) \right) \leq 0.
\]

That is, \(T(u, u, 0, 0, 2u) \leq 0\), hence from \((C_3)\), we have \(u = d(p, gp) = 0\). Hence \(gp = p\). Therefore, \(Gp = \{p\}\). Since \(Gp \subseteq f(X)\), then there exists \(w \in X\) such that \(\{fw\} = Gp = \{gp\} = \{p\}\). Now if set \(w, p\) replacing \(x, y\) respectively, in inequality (iii) we get

\[
T \left( \delta(Fw, Gp), d(fw, gp), D(fw, Fw), D(gp, Gp), D(fw, Gp) + D(gp, Fw) \right) \leq 0.
\]

and so we have

\[
T \left( \delta(Fw, p), 0, \delta(p, Fw), 0, \delta(p, Fw) \right) \leq 0.
\]

That is, \(T(u, 0, u, 0, u) \leq 0\), hence from \((C_3)\), we have \(u = \delta(Fw, p) = 0\). Hence \(Fw = \{p\} = Gp = \{fw\} = \{gp\}\). Since \(Fw = \{fw\}\) and the pair \((F, f)\) is weakly compatible, then we obtain \(Fp = Ffw = FFW = \{fp\}\). Therefore, we obtain \(Fp = Gp = \{fp\} = \{gp\} = \{p\}\).

The proof is similar when \(f(X)\) is assumed to be a closed subset of \(X\).
To see that $p$ is unique, suppose that $\{q\} = \{gq\} = \{fp\} = Fp = Gp$. If $p \neq q$, then
\[ T(\delta(Fp, Gq), d(fp, gq), D(fp, Fp), D(gq, Gq), D(fp, Gq) + D(gq, Fp)) \leq 0, \]
therefore $T(d(p, q), d(p, q), 0, 0, 2d(p, q) \leq 0$, that is $d(p, q) = 0$. It follows that $p = q$. \hfill \Box

**Corollary 1.** Let $F, G$ be mappings of a complete metric space $(X, d)$ into $B(X)$ such that satisfying:

(i) \[ T(\delta(Fx, Gx), d(x, y), D(x, Fx), D(y, Gx), D(x, Gx) + D(y, Fx)) \leq 0 \]
for every $x, y$ in $X$. Then there exists a unique $p \in X$ such that $\{p\} = Fp = Gp$.

**Proof.** By Theorem 1, it is enough defined $f, g$ be identity mappings. \hfill \Box

If we combine Theorem 1 with Example 1 we have the following corollary.

**Corollary 2.** Let $F, G$ be mappings of a complete metric space $(X, d)$ into $B(X)$ and $f, g$ be mappings of $X$ into itself satisfying:

(i) $Fx \subseteq g(X)$, $Gx \subseteq f(X)$ for every $x \in X$,
(ii) The pair $(F, f)$ and $(G, g)$ are weakly compatible,
(iii) $\delta(Fx, Gx) \leq \alpha \max\{d(fx, gy), D(fx, Fx), D(gy, Gx)\} + \beta(D(fx, Gx) + D(gy, Fx))$

for every $x, y$ in $X$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of $X$, then there exists a unique $p \in X$ such that $\{p\} = \{fp\} = \{gp\} = Fp = Gp$.

**Example 3.** Let $X = [0, 1]$ endowed with the Euclidean metric $d$. Define $F, G : X \rightarrow B(X)$ and $f, g : X \rightarrow X$ as follows:

\[
Fx = \{1/2\}, \quad Gx = \begin{cases} 
1/2, & x \in [0, 1/2] \\
3/8, 1/2, & x \in (1/2, 1] 
\end{cases} \\
fx = \begin{cases} 
1/2, & x \in [0, 1/2] \\
x + 1/4, & x \in (1/2, 1] 
\end{cases}, \quad gx = \begin{cases} 
1 - x, & x \in [0, 1/2] \\
0, & x \in (1/2, 1] 
\end{cases}
\]

It is clear that $Fx = \{1/2\} \subseteq g(X) = \{0\} \cup [1/2, 1]$, $Gx = (3/8, 1/2] = f(X)$ and $g(X)$ is closed subset of $X$. Now we consider the following cases:

Case 1. If $x \in [0, 1/2]$ and $y \in [0, 1/2]$, then
\[ \delta(Fx, Gy) = 0 \leq \frac{1}{3} d(fx, gy). \]
Case 2. If \(x \in [0, 1/2]\) and \(y \in (1/2, 1]\), then
\[
\delta(Fx, Gy) = \frac{1}{8} \leq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3} d(fx, gy).
\]

Case 3. If \(x \in (1/2, 1]\) and \(y \in [0, 1/2]\), then
\[
\delta(Fx, Gy) = 0 \leq \frac{1}{3} d(fx, gy).
\]

Case 4. If \(x \in (1/2, 1]\) and \(y \in (1/2, 1]\), then
\[
\delta(Fx, Gy) = \frac{1}{8} \leq \frac{1}{3} \cdot \frac{3}{8} \leq \frac{1}{3} d(fx, gy).
\]

Therefore, we obtain
\[
\delta(Fx, Gy) \leq \frac{1}{3} d(fx, gy)
\]
\[
\leq \frac{1}{3} \max \left\{d(fx, gy), D(fx, Fx), D(gy, Gy), \frac{D(fx, Gy) + D(gy, Fx)}{2}\right\}
\]
for all \(x, y \in X\). That is, the condition (iii) of Theorem 1 is satisfied with
\[
T(t_1, \ldots, t_5) = t_1 - \frac{1}{3} \max \left\{t_2, t_3, t_4, \frac{1}{2} t_5 \right\}.
\]

Also, the coincidence points of \(F\) and \(f\) are \(1/2\) and \(1\), and it is clear that \(F\) and \(f\) are commuting at \(1/2\) and \(1\). Similarly, the only coincidence point of \(G\) and \(g\) is \(1/2\), and \(G\) and \(g\) are commuting at \(1/2\). Thus \(F\) and \(f\) as well as \(G\) and \(g\) are weakly compatible. Consequently all conditions of Theorem 1 are satisfied and so these mappings have a unique common fixed point on \(X\). On the other hand, if \(x_n = \frac{1}{2} - \frac{1}{2n}\), so that \(\delta(Gx_n, gx_n) \to 1/8 \neq 0\) even though \(Gx_n, \{gx_n\} \to \{1/2\}\), that is, the mappings \(G\) and \(g\) are not compatible. Therefore the fixed point results, which have condition of compatibility, are not applicable to this example. For example the results in [6], [8]–[10] and some others.

Acknowledgement. The authors are grateful to the referees for their valuable comments in modifying the first version of this paper.

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