LIMIT DISTRIBUTION OF ASCENT, DESCENT OR EXCEDANCE LENGTH SUMS OF PERMUTATIONS

Lane Clark

Let $A_n(\sigma)$ denote the sum of the lengths of ascents of a permutation $\sigma$ of $\{1, \ldots, n\}$ chosen uniformly at random. We find the exact expectation and variance and prove a central limit theorem for the $A_n$. Identical results hold for the sum of the lengths of descents or of excedances of a permutation of $\{1, \ldots, n\}$ chosen uniformly at random.

1. INTRODUCTION

The set of permutations of $[n] = \{1, \ldots, n\}$ is $S_n = \{(\sigma(1), \ldots, \sigma(n)) : \sigma(1), \ldots, \sigma(n) \in [n] \text{ are distinct}\}$ with equality of tuples. A permutation $\sigma = (\sigma(1), \ldots, \sigma(n)) \in S_n$ has

an ascent at $i \in [n-1]$ iff $\sigma(i) < \sigma(i+1)$ with length of ascent at $i$ of $\sigma(i+1) - \sigma(i)$, 
a descent at $i \in [n-1]$ iff $\sigma(i) > \sigma(i+1)$ with length of descent at $i$ of $\sigma(i) - \sigma(i+1)$, 
an excedance at $i \in [n]$ iff $\sigma(i) > i$ with length of excedance at $i$ of $\sigma(i) - i$.

We consider the uniform probability space $\Omega_n = (S_n, \mathcal{P}_n, \Pr = \Pr_n)$ on $S_n$, i.e., $\mathcal{P}_n$ is the powerset of $S_n$ and $\Pr(\sigma) = 1/n!$ for each $\sigma \in S_n$. Any function $X : S_n \rightarrow \mathbb{N}$ is then a random variable on $\Omega_n$ with finite moments $E(X^r) = \sum_{k=0}^{\infty} k^r \Pr(X = k)$.

Let $A_{n,m}(\sigma)$ denote the number of ascents of $\sigma \in S_n$ of length at least $m \in \mathbb{P}$. Then $A_{n,1}(\sigma)$ counts the number of ascents of $\sigma$ and the Eulerian numbers $A_n(\sigma) = \#\{\sigma \in S_n : A_{n,1}(\sigma) = k\}$. Central limit theorems for the random variables $A_{n,m}$ on $\Omega_n$ were proved by Carlitz et al. [2] for $m = 1$, and, more generally, by the author [3] for $m = o(n)$. Recently, Balcza [1] found the exact expectation and variance for the sum of the lengths of inversions of $\sigma$ on $\Omega_n$.

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Let \( A_n(\sigma) \) (respectively, \( D_n(\sigma) \) and \( E_n(\sigma) \)) denote the sum of the lengths of ascents (respectively, descents and exceedances) of \( \sigma \in \mathfrak{S}_n \). For example, \( \sigma = (4, 1, 7, 10, 6, 3, 8, 2, 9, 5) \) has ascents at 2, 3, 6, 8 of lengths 6, 3, 5, 7; descents at 1, 4, 5, 7, 9 of lengths 3, 4, 3, 6, 4; and exceedances at 1, 3, 4, 5, 7 of lengths 3, 4, 5, 1, 1. Hence, \( A_{10}(\sigma) = 6 + 3 + 5 + 7 = 21 \), \( D_{10}(\sigma) = 3 + 4 + 3 + 6 + 4 = 20 \) and \( E_{10}(\sigma) = 3 + 4 + 6 + 4 + 1 + 1 = 15 \). Evidently the statistic \( A_n(\sigma) \) refines the statistic \( A_{n,1}(\sigma) \). It is easily seen that \( \max\{A_n(\sigma) : \sigma \in \mathfrak{S}_n\} = \max\{D_n(\sigma) : \sigma \in \mathfrak{S}_n\} = \max\{E_n(\sigma) : \sigma \in \mathfrak{S}_n\} = M_n \) (\( n \in \mathbb{P} \)) where \( M_n = [n/2]^2 \) (even \( n \)) and \( M_n = [n/2]^2 - [n/2] \) (odd \( n \)).

Let \( a(n,k) = \#\{\sigma \in \mathfrak{S}_n : A_n(\sigma) = k\} \), \( d(n,k) = \#\{\sigma \in \mathfrak{S}_n : D_n(\sigma) = k\} \) and \( e(n,k) = \#\{\sigma \in \mathfrak{S}_n : E_n(\sigma) = k\} \) (see Table 1 below). Then \( A_n, D_n \) and \( E_n \) are random variables on \( \Omega_n \) with

\[
\Pr(A_n = k) = \frac{a(n,k)}{n!}, \quad \Pr(D_n = k) = \frac{d(n,k)}{n!} \quad \text{and} \quad \Pr(E_n = k) = \frac{e(n,k)}{n!}. \quad (k \in \mathbb{N})
\]

It is clear that \( a(n,k) = d(n,k) \) \( (n \in \mathbb{P}, k \in \mathbb{N}) \) by just reading the permutations in the opposite direction. Lemma 2.1 proves that \( a(n,k) = e(n,k) \) \( (n \in \mathbb{P}, k \in \mathbb{N}) \). Hence, \( A_n, D_n \) and \( E_n \) are identically distributed (they are not pair-wise independent, however) on \( \Omega_n \). Therefore, from here on we let \{\( X_n \)\} \( = \{A_n\}, \{D_n\} \) or \{\( E_n\)\} on \( \Omega_n \). In this paper, we derive \( \mu_n = E(X_n) = (n^2 - 1)/6 \) and \( \sigma_n^2 = \text{Var}(X_n) = (2n^3 + 2n^2 + 7n + 7)/180 \) and prove the central limit theorem \( (X_n - \mu_n)/\sigma_n \rightarrow N(0,1) \), i.e, we prove that for every \( x \in \mathbb{R} \)

\[
\Pr \left( \frac{X_n - \mu_n}{\sigma_n} \leq x \right) \rightarrow \Phi(x),
\]

where

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt
\]

is the distribution function of a standard normal random variable \( N(0,1) \).

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\( a(n,k) \)

Table 1.

In passing we note that the entries in Table 1, read by rows, is The Online Encyclopedia of Integer Sequences sequence A062869 (date: June 26, 2001) about
2.1. Equidistribution of $\sigma$

is the largest integer at most $x$ respectively, the positive integers, the rational numbers and the real numbers). Also $\lfloor x \rfloor$ is the largest integer at most $x \in \mathbb{R}$. Let $(r)_k = 1$ and $(r)_k = (r) \cdots (r-k+1)$ ($k \in \mathbb{P}, r \in \mathbb{R}$). See Comtet [4] for combinatorics and Durrett [5] for probability.

2. MAIN RESULTS

2.1. Equidistribution of $A_n$, $D_n$ and $E_n$

Stanley [8, Proposition 1.3.12] gave an explicit bijection $f : \mathfrak{S}_n \to \mathfrak{S}_n$ so that the number of ascents of $\sigma$ equals the number of excedances of $f(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. We next give a different bijection $f$ that also satisfies $A_n(\sigma) = E_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

Lemma 2.1. The function $f : \mathfrak{S}_n \to \mathfrak{S}_n$ defined in the proof is a bijection where the number of ascents of $\sigma$ equals the number of excedances of $f(\sigma)$ and $A_n(\sigma) = E_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$. Hence, $a(n,k) = c(n,k)$ for all $n \in \mathbb{P}$ and $k \in \mathbb{N}$.

Proof. Suppose $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$ has ascents at $1 \leq i_1 < \cdots < i_\ell \leq n-1$. Order $\sigma(i_1), \ldots, \sigma(i_\ell)$ as $1 \leq \sigma(j_1) < \cdots < \sigma(j_\ell) \leq n$ and $[n] - \{\sigma(j_\ell + 1) : 1 \leq k \leq \ell\}$ as $1 \leq t_1 < \cdots < t_{n-\ell} \leq n$. Now construct $\tau \in \mathfrak{S}_n$ as follows. Place $\sigma(j_k + 1)$ at coordinate $\sigma(j_k)$ of $\tau$ ($1 \leq k \leq \ell$). Necessarily, $t_1 = 1$ (as all $\sigma(j_k + 1) \geq \sigma(j_k) + 1 \geq 2$) which we place in the left-most unused coordinate $s_1$ of $\tau$. Having placed $t_1, \ldots, t_q$ in (the left-most unused) coordinates $1 \leq s_1 < \cdots < s_q$, we place $t_{q+1}$ in the left-most unused coordinate $s_{q+1}$ (greater than $s_q$, necessarily) of $\tau$ ($1 \leq q \leq n - \ell - 1$). Clearly, $\tau \in \mathfrak{S}_n$.

Assume that all of $1, \ldots, t_q$ have appeared in coordinates $1, \ldots, s_q$ of $\tau$ where $1 \leq q \leq n - \ell - 1$. Let $s_{q+1} = s_q + a$, $t_{q+1} = t_q + b$ and $s_q = s_q + c$ with $a, b \in \mathbb{N}$ and $c \in \mathbb{N}$. Suppose that $t_{q+1} \geq s_{q+1} + 1$. For $1 \leq x \leq a + c$, $t_q + 1 \leq t_q + x \leq t_q + a + c = s_q + 1 \leq t_{q+1} - 1$. Then, each $t_q + x = \sigma(j_k + 1)$ is at coordinate $\sigma(j_k)$ with $\sigma(j_k) \leq t_q + x - 1 \leq t_q + a + c - 1 = s_q + 1 - 1$. Hence, all of $t_{q+1}, \ldots, t_q + a + c = s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$. Consequently, all of $1, \ldots, s_{q+1}$ appear in coordinates $1, \ldots, s_{q+1} - 1$, which is a contradiction. Then $t_{q+1} \leq s_{q+1}$ and, as above, all of $t_q + 1, \ldots, t_{q+1}$ appear in coordinates $1, \ldots, s_{q+1}$. Hence, all of $1, \ldots, t_{q+1}$ appear in coordinates $1, \ldots, s_{q+1}$.

Consequently, $s_q \geq t_q$ ($1 \leq q \leq n - \ell$), so that the number of ascents of $\sigma$ equals the number of excedances of $\tau$ and $A_n(\sigma) = E_n(\tau)$. It is immediately seen that $f : \mathfrak{S}_n \to \mathfrak{S}_n$ by $f : \sigma \mapsto \tau$ is a bijection proving our result.

2.2. Expectation and Variance of $X_n$

Theorem 2.2. The random variables $X_n$ on $\Omega_n$ have

$$
\mu_n = E(X_n) = \frac{n^2 - 1}{6} \quad \text{and} \quad \sigma_n^2 = \text{Var}(X_n) = \frac{2n^3 + 2n^2 + 7n + 7}{180}.
$$
Proof. We prove the theorem for the statistic $A_n$. Let $I = \{(i, r, s) : 1 \leq i \leq n - 1, 1 \leq r < s \leq n\}$. For $(i, r, s) \in I$ and $\sigma \in \mathfrak{S}_n$, let

$$A_{(i, r, s)}(\sigma) = \begin{cases} s - r, & \sigma(i) = r, \sigma(i + 1) = s; \\ 0, & \text{otherwise,} \end{cases}$$

and $A_n = \sum_{(i, r, s) \in I} A_{(i, r, s)}$. Then $A_n(\sigma)$ is the sum of the lengths of ascents of $\sigma$.

For $(i, r, s) \in I$, $E(A_{(i, r, s)}) = \frac{s - r}{(n)_2}$, hence,

$$E(A_n) = \sum_{(i, r, s) \in I} E(A_{(i, r, s)}) = \sum_{i = 1}^{n - 1} \sum_{s = i + 1}^{n} \frac{s - r}{(n)_2} = \frac{n^2 - 1}{6}.$$

Let $J = I^2 - \{(i, r, s), (i, r, s) : (i, r, s) \in I\}$. Then,

$$A_n^2 = \sum_{(i, r, s), (j, t, u) \in I^2} A_{(i, r, s)}A_{(j, t, u)} = \sum_{(i, r, s) \in I} A_{(i, r, s)}^2 + \sum_{(i, r, s), (j, t, u) \in J} A_{(i, r, s)}A_{(j, t, u)}.$$

Obviously $E(A_{(i, r, s)}A_{(j, t, u)}) = E(A_{(j, t, u)}A_{(i, r, s)})$ for $((i, r, s), (j, t, u)) \in I^2$. Further, $E(A_{(i, r, s)}A_{(j, t, u)}) = 0$ if (1) $j = i + 1$ and $s \neq t$, (2) $i = j + 1$ and $r \neq u$, or, (3) $|i - j| \geq 2$ and $r, s, t, u$ are distinct. Hence, we need only consider the sets

$J_1 = \{(i, r, s), (i + 1, s, t) \in J : 1 \leq i \leq n - 2\}, \quad J_1^*$

$J_2 = \{(i, r, s), (j, t, u) \in J : 1 \leq i \leq j - 2 \leq n - 3 \text{ and } r, s, t, u \text{ are distinct}\}, \quad J_2^*$

where $K^* = \{(b, a) : (a, b) \in K\}$ for $K \subseteq J$. First, for $(i, r, s) \in I$, $E(A_{(i, r, s)}^2) = \frac{(s - r)^2}{(n)_2}$, hence,

$$\Sigma_1 := \sum_{(i, r, s) \in I} E(A_{(i, r, s)}^2) = \sum_{i = 1}^{n - 1} \sum_{s = i + 1}^{n} \frac{(s - r)^2}{(n)_2} = \frac{n^3 - n}{12}.$$

Second, for $((i, r, s), (i + 1, s, t)) \in J_1$, $E(A_{(i, r, s)}A_{(i + 1, s, t)}) = \frac{(s - r)(t - s)}{(n)_3}$, hence,

$$\Sigma_2 := \sum_{((i, r, s), (i + 1, s, t)) \in J_1} E(A_{(i, r, s)}A_{(i + 1, s, t)}) = \sum_{i = 1}^{n - 2} \sum_{t = 3}^{n} \sum_{s = 2}^{n - 1} \frac{(s - r)(t - s)}{(n)_3}$$

$$= \frac{1}{2(n)_3} \sum_{i = 1}^{n - 2} \sum_{t = 3}^{n} \sum_{s = 1}^{t - 1} \{-s^3 + (t + 1)s^2 - ts\}$$

$$= \frac{1}{24(n)_3} \sum_{i = 1}^{n - 2} \sum_{t = 1}^{n} \{t^4 - 2t^3 - t^2 + 2t\}$$

$$= \frac{1}{120(n)_3} \sum_{i = 1}^{n - 2} \{n^5 - 5n^3 + 4n\} = \frac{n^5 + n^2 - 4n - 4}{120}$$. 

(appropriate summands for $s = 1$ and $t = 1, 2$ are 0). Third, for $((i, r, s), (j, t, u)) \in J_2$, $E(A_{(i, r, s)}A_{(j, t, u)}) = \frac{(s - r)(u - t)}{(n)_4}$. For fixed $1 \leq i \leq j - 2 \leq n - 3$,

$$\Sigma_3 := \sum_{\text{all such } ((i, r, s), (j, t, u))} E(A_{(i, r, s)}A_{(j, t, u)}) = \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s - r) \left( \sum_{u=2}^{n} \sum_{t=1}^{u-1} (u - t) - \sum_{t=1}^{r-1} (r - t) - \sum_{t=1}^{s-1} (s - t) \right)$$

$$= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s - r) \left( \sum_{u=r+1}^{n} (u - r) - \sum_{u=s+1}^{n} (u - s) + (s - r) \right)$$

$$= \frac{1}{(n)_4} \sum_{s=2}^{n} \sum_{r=1}^{s-1} (s - r) \left\{ \binom{n+1}{3} + (s - r) - \{r^2 - r + s^2 - s + n^2 - (r + s - 1)n\} \right\}.$$

Now,

$$\sum_{s=2}^{n} \sum_{r=1}^{s-1} (s - r)\{(r^2 - r + s^2 - s + n^2 - (r + s - 1)n\}

= \sum_{s=2}^{n} \sum_{r=1}^{s-1} \left\{ -r^3 + (n + 1 + s)r^2 - (n^2 + n + 2)s^3 \right\}$$

$$= \frac{1}{12} \sum_{s=1}^{n} \left\{ 7s^4 - (8n + 14)s^3 + (6n^2 + 12n + 5)s^2 - (6n^2 + 4n - 2)s \right\} = \frac{7n^5 - 15n^3 + 8n}{60}.$$

Then,

$$\Sigma_4 = \frac{1}{(n)_4} \left\{ \binom{n+1}{3} + \frac{n^4 - n^2}{12} - \frac{7n^5 - 15n^3 + 8n}{60} \right\}$$

$$= \frac{10n^5 - 42n^4 + 10n^3 + 90n^2 - 20n - 48}{360(n - 1)^3},$$

hence, summing over all such $i$ and $j$,

$$\Sigma_4 := \sum_{((i, r, s), (j, t, u)) \in J_2} E(A_{(i, r, s)}A_{(j, t, u)})$$

$$= \binom{n - 2}{2} \Sigma_3 = \frac{5n^4 - 16n^3 - 11n^2 + 34n + 24}{360}. $$
Proof. We prove the theorem for the statistic $d_n$. Equivalently, $a_n + 1 \leq n$. All $d_n(i, j) \leq d_n(1, n) = (n^2 - 1)/6n < n/6$. Set $a = [2n/3]$ and $b = [n/3]$. For $a + 1 \leq i \leq n$, $1 \leq j \leq b$ and $n \geq 88$, $d_n(i, j) \geq n/20$. Then $\sum_{i=1}^{n} \sum_{j=1}^{n} d_n^2(i, j) \geq \frac{2n^3 + 2n^2 + 7n + 7}{180}$. For $b < i \leq n$, $1 \leq j \leq b$ and $b \leq j \leq n$, $d_n(i, j) \geq \frac{2n^3 + 2n^2 + 7n + 7}{180}$. Consequently, $E(A_n^2) = \sum_{i=1}^{n} 2\sum_{j=1}^{n} 2\sum_{k=1}^{n} \frac{5n^4 + 2n^3 - 8n^2 + 7n + 12}{180}$, hence, $\text{Var}(A_n) = \frac{2n^3 + 2n^2 + 7n + 7}{180}$. \hfill \Box

2.3. Central Limit Theorem for $\{X_n\}$

We now prove a central limit theorem for $\{X_n\}$. Our proof is based on the following result of Hoeffding [7; Theorem 3]. Given $c_n : [n]^2 \to \mathbb{R}$, let $d_n : [n]^2 \to \mathbb{R}$ be defined by $d_n(i, j) = c_n(i, j) - \frac{1}{n} \sum_{g=1}^{n} c_n(g, j) - \frac{1}{n} \sum_{h=1}^{n} c_n(i, h) + \frac{1}{n^2} \sum_{g=1}^{n} \sum_{h=1}^{n} c_n(g, h).

Theorem 2.3 (Hoeffding [7]). Suppose random variables $S_n : \mathbb{N} \to \mathbb{R}$ on $\Omega_n$ are defined by $S_n(\sigma) = \sum_{i=1}^{n} c_n(i, \sigma(i))$. If $\lim_{n \to \infty} \frac{\max_{1 \leq i, j \leq n} d_n^2(i, j)}{\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} d_n^2(i, j)} = 0$, then $\frac{S_n - \mu_n}{\sigma_n} \overset{d}{\to} N(0, 1)\text{ for } N(0, 1)$ where $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} c_n(i, j)$ and $\sigma_n = \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{n} d_n^2(i, j)$.

Theorem 2.4. The random variables $X_n$ on $\Omega_n$ satisfy the central limit theorem

$$\frac{X_n - \mu_n}{\sigma_n} \overset{d}{\to} N(0, 1)$$

where $\mu_n = E(X_n) = (n^2 - 1)/6$ and $\sigma_n^2 = \text{Var}(X_n) = (2n^3 + 2n^2 + 7n + 7)/180$. Equivalently (see Durrett [5; Ex. 2.1, p.70]),

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \Pr(X_n \leq \mu_n + x\sigma_n) - \Phi(x) \right| = 0.$$ 

Proof. We prove the theorem for the statistic $E_n$. Let $c_n : [n]^2 \to \mathbb{N}$ be defined by $c_n(i, j) = \max\{0, j - i\}$. From Theorem 2.3, $S_n(\sigma) = \sum_{i=1}^{n} c_n(i, \sigma(i)) = E_n(\sigma)$ and $d_n : [n]^2 \to \mathbb{Q}$ is given by

$$d_n(i, j) = \begin{cases} j - i - \frac{1}{n} \binom{j}{2} - \frac{1}{n} \binom{n - i + 1}{2} + \frac{1}{n^2} \binom{n + 1}{3}, & 1 \leq i < j \leq n; \\ \frac{1}{n} \binom{j}{2} - \frac{1}{n} \binom{n - i + 1}{2} + \frac{1}{n^2} \binom{n + 1}{3}, & 1 \leq j \leq i \leq n. \end{cases}$$

All $d_n(i, j) \leq d_n(1, n) = (n^2 - 1)/6n < n/6$. Set $a = [2n/3]$ and $b = [n/3]$. For $a + 1 \leq i \leq n$, $1 \leq j \leq b$ and $n \geq 88$, $d_n(i, j) \geq n/20$. Then $\sum_{i=1}^{n} \sum_{j=1}^{n} d_n^2(i, j) \geq \frac{2n^3 + 2n^2 + 7n + 7}{180}$. For $b < i \leq n$, $1 \leq j \leq b$ and $b \leq j \leq n$, $d_n(i, j) \geq \frac{2n^3 + 2n^2 + 7n + 7}{180}$. Consequently, $E(A_n^2) = \sum_{i=1}^{n} 2\sum_{j=1}^{n} 2\sum_{k=1}^{n} \frac{5n^4 + 2n^3 - 8n^2 + 7n + 12}{180}$, hence, $\text{Var}(A_n) = \frac{2n^3 + 2n^2 + 7n + 7}{180}$. \hfill \Box
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\[ \sum_{i=a+1}^{n} \sum_{j=1}^{b} d_{2}^{2}(i, j) \geq \frac{n^{4}}{4000}, \text{ hence, } \frac{\max_{1 \leq i \leq n} d_{2}^{2}(i, j)}{\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{2}^{2}(i, j)} = O(n^{-1}). \] Lemma 2.1 implies the \( \mu_{n} \) and \( \sigma_{n} \) of Theorems 2.2 and 2.3 are identical (as can be verified). Our result follows from Theorem 2.3.

Theorem 2.4 and (1) imply the following asymptotic result.

**Corollary 2.5.** With \( \mu_{n} = \left( n^{2} - 1 \right)/6 \) and \( \sigma_{n}^{2} = (2n^{3} + 2n^{2} + 7n + 7)/180, \)

\[ \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{n!} \sum_{k=0}^{[\mu_{n} + x\sigma_{n}]} a(n, k) - \Phi(x) \right| = 0. \]

In particular, \( \sum_{k=[\mu_{n} + \alpha\sigma_{n}]}^{[\mu_{n} + \beta\sigma_{n}]} a(n, k) \sim \{ \Phi(\beta) - \Phi(\alpha) \} \ n! \) uniformly for real \( \alpha < \beta. \)

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