SOME FORMULAS FOR APOSTOL-EULER POLYNOMIALS ASSOCIATED WITH HURWITZ ZETA FUNCTION AT RATIONAL ARGUMENTS

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We give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and obtain some series representations of the Apostol-Euler polynomials of higher order in terms of the generalized Hurwitz-Lerch Zeta function. Several interesting special cases are also shown.

1. INTRODUCTION

Throughout this paper, we always make use of the following notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0 = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha$ (real or complex) are usually defined by means of the following generating functions (see, for details, [1, 5, 13, 15]):

\[ \left( \frac{z}{e^z - 1} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < 2\pi) \]

and

\[ \left( \frac{2}{e^z + 1} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{z^n}{n!} \quad (|z| < \pi) \].

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Obviously, the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by
\[(1.3) \quad B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0),\]
respectively. The classical Bernoulli numbers $B_n$ and Euler numbers $E_n$ are defined by
\[(1.4) \quad B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n \left( \frac{1}{2} \right) \quad (n \in \mathbb{N}_0),\]
respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol \cite{2, p. 165, Eq. (3.1)} and (more recently) by Srivastava \cite{14, p. 83–84}. We begin by recalling Apostol’s definitions as follows:

**Definition 1.1 (Apostol \cite{2}; see also Srivastava \cite{14}).** The Apostol-Bernoulli polynomials $B_n(x; \lambda)$ in $x$ are defined by means of the generating function:
\[(1.5) \quad \frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{z^n}{n!} \quad (|z| < 2\pi \text{ when } \lambda = 1; \quad |z| < |\log \lambda| \text{ when } \lambda \neq 1)\]
with, of course,
\[(1.6) \quad B_n(x) = B_n(x;1) \quad \text{and} \quad B_n(\lambda) := B_n(0; \lambda),\]
where $B_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (in fact, it is a function in $\lambda$).

Recently, Luo and Srivastava extended further the Apostol-Bernoulli and Apostol-Euler polynomials and their generalizations as follows:

**Definition 1.2 (cf. Luo and Srivastava \cite{10, 12}).** The Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ of order $\alpha$ are defined by means of the generating function
\[(1.7) \quad \left( \frac{ze^{xz}}{\lambda e^z - 1} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!} \quad (|z| < 2\pi \text{ when } \lambda = 1; \quad |z| < |\log \lambda| \text{ when } \lambda \neq 1)\]
with, of course,
\[(1.8) \quad B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1) \quad \text{and} \quad B_n^{(\alpha)}(\lambda) := B_n^{(\alpha)}(0; \lambda),\]
\[B_n(x; \lambda) := B_n^{(1)}(x; \lambda) \quad \text{and} \quad B_n(\lambda) := B_n(0; \lambda),\]
where $B_n(\lambda)$, $B_n^{(\alpha)}(\lambda)$ and $B_n(x; \lambda)$ denote the so-called Apostol-Bernoulli numbers, Apostol-Bernoulli numbers of order $\alpha$ and Apostol-Bernoulli polynomials respectively.
Definition 1.3 (cf. Luo [11]). The Apostol-Euler polynomials $E_n^{(\alpha)} (x; \lambda)$ of order $\alpha$ are defined by means of the generating function

$$
(1.9) \quad \left( \frac{2}{\lambda e^z + 1} \right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)} (x; \lambda) \frac{z^n}{n!} \quad (|z| < |\log(-\lambda)|),
$$

with, of course,

$$
(1.10) \quad E_n^{(\alpha)} (x) = E_n^{(\alpha)} (x; 1) \quad \text{and} \quad E_n^{(\alpha)} (\lambda) := 2^n E_n^{(\alpha)} \left( \frac{\alpha}{2}; \lambda \right),
$$

where $E_n (\lambda)$, $E_n^{(\alpha)} (\lambda)$ and $E_n (x; \lambda)$ denote the so-called Apostol-Euler numbers, Apostol-Euler numbers of order $\alpha$ and Apostol-Euler polynomials respectively.

The main object of the present paper is to give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and to investigate some series representations of the Apostol-Euler polynomials in terms of generalized Hurwitz-Lerch Zeta function.

2. SOME EXPLICIT RELATIONSHIPS BETWEEN THE APOSTOL-EULER POLYNOMIALS AND THE GENERALIZED HURWITZ-LERCH ZETA FUNCTION

A family of the Hurwitz-Lerch Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)} (z, s, a)$ defined by (see e.g. [9, p. 727, Eq. (8)]

$$
(2.1) \quad \Phi_{\mu,\nu}^{(\rho,\sigma)} (z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n + a)^s}
$$

(\mu, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho, \sigma \in \mathbb{R}^+; \rho < \sigma \quad \text{when} \quad s, z \in \mathbb{C}; \rho = \sigma \quad \text{and} \quad s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \rho = \sigma \quad \text{and} \quad \Re(s - \mu + \nu) > 1 \quad \text{when} \quad |z| = 1),
$$

contains, as its special cases, not only the Hurwitz-Lerch Zeta function

$$
(2.2) \quad \Phi_{\mu,\nu}^{(\sigma,\sigma)} (z, s, a) = \Phi_{\mu,\nu}^{(0,0)} (z, s, a) = \Phi (z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s}
$$

and the Lipschitz-Lerch Zeta function (cf. [15, p. 122, Eq. 2.5 (11)]):

$$
(2.3) \quad \phi (\xi, a, s) := \Phi \left( e^{2\pi i \xi}; s, a \right) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \xi}}{(n + a)^s}
$$

(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z}; \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}),
$$
but also the following generalized Hurwitz-Lerch Zeta functions introduced and studied earlier by Goyal and Laddha [7, p. 100, Eq. (1.5)]

\[(2.4) \Phi_{\mu,1}^{(1,1)}(z, s, a) = \Phi_{\mu}(z, s, a) := \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{z^n}{(n+a)^s},\]

which, for convenience, are called the Goyal-Laddha-Hurwitz-Lerch Zeta function. Here the symbol \((a)_k\) denotes the Pochhammer symbol or the shifted factorial defined, \(a \in \mathbb{C}\), by

\[(2.5) (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} 1 & (k = 0), \\ a(a+1) \cdots (a+k-1) & (k \in \mathbb{N}), \end{cases}\]

where \(\Gamma(x)\) is the usual Gamma function.

Recently, Garg et al. [6] obtained the following interesting formula:

\[(2.6) B^\ell_n(a; \lambda) = (-n)^\ell \Phi_\ell(-\lambda, \ell - n, a) \quad (n, \ell \in \mathbb{N}; \ n \geq \ell; \ |\lambda| < 1; \ a \in \mathbb{C} \setminus \mathbb{Z}^-_0).\]

Clearly, we have

\[(2.7) B_n(a; \lambda) = -n \Phi(-\lambda, 1 - n, a) \quad (n \in \mathbb{N}; \ |\lambda| \leq 1; \ a \in \mathbb{C} \setminus \mathbb{Z}^-_0).\]

Below we give the following explicit relationships between the a family of Euler polynomials and a family of Zeta function.

**Theorem 2.1.** \(\text{For } n \in \mathbb{N}; -1 < \lambda \leq 1; \ a \in \mathbb{C}; \ a \in \mathbb{C} \setminus \mathbb{Z}^-_0, \) the following relationship

\[(2.8) E_n^{(\alpha)}(a; \lambda) = 2^\alpha \Phi_n(-\lambda, -n, a)\]

between the Apostol-Euler polynomials of higher order and the Goyal-Laddha-Hurwitz-Lerch Zeta function.

**Proof.** By (1.9) and the generalized binomial theorem, yields

\[(2.9) \sum_{n=0}^{\infty} E_n^{(\alpha)}(a; \lambda) \frac{z^n}{n!} = \left(\frac{2}{\alpha e^z + 1}\right)^\alpha e^{az} = 2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} (-\lambda)^k e^{(k+a)z} = \sum_{n=0}^{\infty} \left[ 2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} (-\lambda)^k (k+a)^n \right] \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left[ 2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{(-\lambda)^k}{(k+a)^n} \right] \frac{z^n}{n!}.

Hence, the formula (2.8) follows. \(\square\)

**Corollary 2.2.** \(\text{For } n \in \mathbb{N}; -1 < \lambda \leq 1; \ a \in \mathbb{C} \setminus \mathbb{Z}^-_0, \) the following relationship

\[(2.10) E_n(a; \lambda) = 2\Phi(-\lambda, -n, a)\]
holds true between the Apostol-Euler polynomials and the Hurwitz-Lerch Zeta function.

It is well-known that the following relationship between the Bernoulli polynomials and the Hurwitz Zeta function (see Apostol [3, p. 264, Theorem 12.13])

\[
B_n(a) = -n\zeta(1-n, a) \quad (n \in \mathbb{N}),
\]

where \(\zeta(s, a)\) denotes the Hurwitz Zeta function defined by

\[
\zeta(s, a) := \Phi(1, s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \quad (\Re(s) > 1; \quad a \in \mathbb{C} \setminus \mathbb{Z}_{<0}).
\]

An alternating series version of the Hurwitz Zeta function is given as follows:

**Definition 2.3.** The \(L\)-function is defined by

\[
L(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s} \quad (\Re(s) > 1; \quad a \in \mathbb{C} \setminus \mathbb{Z}_{<0}).
\]

In the same method, it is not difficult, we give a quasi formula of (2.11) as follows:

**Theorem 2.4.** For \(n \in \mathbb{N}; \quad a \in \mathbb{C} \setminus \mathbb{Z}_{<0}\), the following relationship

\[
E_n(a) = 2L(-n, a)
\]

holds true between the Euler polynomials and the \(L\)-function.

It is well-known that the following relationship between the Bernoulli numbers and the Riemann Zeta function (see [3, p. 266, Theorem 12.16])

\[
B_n = -n\zeta(1-n) \quad (n \in \mathbb{N}),
\]

where \(\zeta(s)\) denotes the Riemann Zeta function defined by

\[
\zeta(s) := \zeta(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

An alternating series version of the Riemann Zeta function is given as follows:

**Definition 2.5.** For \(\Re(s) > 0\), the \(\ell\)-function is defined by

\[
\ell(s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.
\]

Similarly, we give an analogue of the formula (2.14) as follows:

**Theorem 2.6.** For \(n \in \mathbb{N}\), the following relationship

\[
E_n = 2\ell(-n)
\]

holds true between the Euler numbers and the \(\ell\)-function.
3. EXPLICIT SERIES REPRESENTATIONS FOR THE
APOSTOL-EULER POLYNOMIALS OF ORDER $\alpha$

It is not difficult, we make use of the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \sum_{k=0}^{\infty} f(qk + j), \quad (q \in \mathbb{N}),$$

to the Hurwitz-Lerch Zeta function (2.2), yields that

$$\Phi(z, s, a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a + j - 1}{q}\right) z^{j-1}.$$ 

Obviously, a special case of (3.2) when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \quad (p \in \mathbb{Z}, \ q \in \mathbb{N})$$

is the following summation formula for the Lipschitz-Lerch Zeta function $\phi(\xi, a, s)$ defined by (2.3):

$$\Phi\left(\exp\left(\frac{2p\pi i}{q}\right), s, a\right) = \phi\left(\frac{p}{q}, a, s\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a + j - 1}{q}\right) \exp\left(\frac{2(j - 1)p\pi i}{q}\right),$$

in terms of the Hurwitz Zeta function $\zeta(s, a)$.

**Theorem 3.1.** For $n, q \in \mathbb{N}; \ p \in \mathbb{Z}; \ \xi \in \mathbb{R}, \ \alpha \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order $\alpha$

$$E_n^{(\alpha)}\left(p, q; e^{2\pi i \xi}\right) = \frac{i^{2\alpha} \cdot n!}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \left(\frac{\alpha - 1}{k}\right) \left(\frac{k - \frac{p}{q} + 1}{q}\right)_{\alpha - k - 1}$$

$$\times \sum_{j=0}^{k} \binom{k - 1}{j - 1} (n + 1)_j B_{k-j}^{(k)}$$

$$\times (2\pi q)^{-n-j-1} \left\{ \sum_{r=1}^{q} \zeta\left(n + j + 1, \frac{2\xi + 2r - 1}{2q}\right) \right\}$$

$$\times \exp\left[\left(\frac{n + j}{2} - \frac{(2\xi + 2r - 1)p}{2}\right) \pi i\right] - \sum_{r=1}^{q} \zeta\left(n + j + 1, \frac{2r - 2\xi + 1}{2q}\right)$$

$$\times \exp\left[\left(- \frac{n + j}{2} + \frac{2r - 2\xi + 1}{2}\right) \pi i\right].$$
holds true in terms of the Hurwitz Zeta function.

**Proof.** We now rewrite the result of Lin et al. as follows (see [8, p. 823, Theorem]):

\[
\Phi_{\mu}(z, s, a) = iz^{a} \Gamma(1-s) \sum_{k=0}^{\infty} \frac{(k-a+1)_{\mu-k-1}}{k! \Gamma(\mu-k)} \\
\times \sum_{j=0}^{k} \frac{(k-1)(1-s)_{j} B_{k-j}^{(j)} (2\pi)^{s-j-1}}{\Gamma(1-s+j, \frac{\log z}{2\pi i})} \\
- \exp\left[\left(2a + \frac{1}{2}(s-j)\right) \pi i\right] \Phi\left(e^{2\pi ai}, 1-s+j, 1 - \frac{\log z}{2\pi i}\right) \quad (\mu \in \mathbb{C}).
\]

Setting \(z = -e^{2\pi i \xi}, \quad a = \frac{p}{q}\) and \(s \mapsto -s, \quad \mu \mapsto \alpha\)

and by applying the series identity (3.3), we find that

\[
\Phi_{\alpha}\left(-e^{2\pi i \xi}, -s, \frac{p}{q}\right) = \frac{i \Gamma(s+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\alpha-1}{k} \left(k - \frac{p}{q} + 1\right)^{\alpha-k-1} \\
\times \left(\sum_{j=0}^{k} \frac{(k-1)(s+1)_{j} B_{k-j}^{(j)}}{(2\pi q)^{s-j-1}} \left(\sum_{r=1}^{q} \zeta\left(s+j+1, \frac{2\xi+2r-1}{2q}\right) - \sum_{r=1}^{q} \zeta\left(s+j+1, \frac{2r-2\xi+1}{2q}\right) \pi i \right) \right) \quad (\alpha \in \mathbb{C}).
\]

Taking \(s = n\) in (3.6) and noting that (2.8) of Theorem 2.1, of course, with \(\lambda = e^{2\pi i \xi}\) and \(a = \frac{p}{q}\),

we obtain the desire (3.4). This proof is complete. \(\square\)

**Theorem 3.2.** For \(n, q, \ell \in \mathbb{N}; \quad p \in \mathbb{Z}; \quad \xi \in \mathbb{C},\) the following formula of the Apostol-Euler polynomials of order \(\ell\)

\[
E^{(\ell)}_{n}\left(\frac{p}{q} ; e^{2\pi i \xi}\right) = \frac{i 2^{\ell} \cdot n!}{(\ell - 1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} \left(k - \frac{p}{q} + 1\right)_{\ell-k-1}
\]
\[ \times \sum_{j=0}^{k} \binom{k-1}{j} (n+1)_j B_k^{(k)} \]
\[ \times (2\pi q)^{-n-j-1} \left\{ \sum_{r=1}^{q} \zeta \left( n+j+1, \frac{2r-2\xi+1}{2q} \right) \right\} \]
\[ \times \exp \left[ \left( \frac{n+j}{2} - \frac{2\xi+2r-1}{2q} \right) \pi i \right] - \sum_{r=1}^{q} \zeta \left( n+j+1, \frac{2r-2\xi}{2q} \right) \]
\[ \times \exp \left[ \left( \frac{n+j}{2} + \frac{2r-2\xi+1}{2q} \right) \pi i \right] \]
holds true in terms of the Hurwitz Zeta function.

**Proof.** Let \( \alpha = \ell (\ell \in \mathbb{N}) \) in (3.6) we may obtain the assertion (3.7). \( \square \)

**Theorem 3.3.** For \( n, q, \ell \in \mathbb{N}; \, p \in \mathbb{Z}; \, \xi \in \mathbb{C} \), the following formula of the Apostol-Euler polynomials of order \( \ell \)
\[ (3.8) \quad E_n^{(\ell)} \left( \frac{p}{q}, e^{2\pi i \xi} \right) = -\frac{i(-2)^{\ell} \cdot n!}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} B_{k-1-q}^{(\ell)} n-k-1 \]
\[ \times \sum_{j=0}^{k} \binom{k}{j} (-n-1) j! p^{-j} (2\pi)^{-n-j-1} \]
\[ \times \left\{ \sum_{r=1}^{q} \zeta \left( n+j+1, \frac{2r+2\xi-1}{2q} \right) \exp \left[ \left( \frac{n+j}{2} - \frac{2\xi+2r-1}{2q} \right) \pi i \right] \right\} \]
\[ - \sum_{r=1}^{q} \zeta \left( n+j+1, \frac{2r-2\xi+1}{2q} \right) \exp \left[ \left( \frac{n+j}{2} + \frac{2r-2\xi+1}{2q} \right) \pi i \right] \]
holds true in terms of the Hurwitz Zeta function.

**Proof.** Setting \( \mu = m (m \in \mathbb{N}) \) in (3.5), we obtain the following transformation formula:
\[ (3.9) \quad \Phi_m(z, s, a) = \frac{i z^{-a} \Gamma(1-s)}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} B_{m-k-1}^{(m)} \]
\[ \times \sum_{j=0}^{k} (-1)^{m-k+j-1} \binom{k}{j} (s-1) j! (-a)^{k-j} (2\pi)^{s-j-1} \]
\[ \times \left[ \exp \left( \frac{1}{2} (s-j) \pi i \right) \Phi \left( e^{-2\pi i \xi}, 1-s+j, \log \frac{z}{2\pi i} \right) \right] \]
\[ - \exp \left[ \left( 2a + \frac{1}{2} (s-j) \right) \pi i \right] \Phi \left( e^{2\pi i \xi}, 1-s+j, 1-\frac{\log z}{2\pi i} \right) \quad (m \in \mathbb{N}). \]
Letting 
\[ z = -e^{2\pi i \xi}, \quad a = \frac{p}{q} \] 
and by applying the series identity (3.3), we obtain the following consequence

\[ \Phi_{\ell} \left( -e^{2\pi i \xi}, -s, -\frac{p}{q} \right) = \frac{i(-1)^{\ell-1} \Gamma(s+1)}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} B_{\ell-k-1} q^{-s-k-1} \]

\[ \times \sum_{j=0}^{k} \binom{k}{j} -s-1 j j! \left( p^{k-j} (2\pi)^{-s-j-1} \right) \]

\[ \times \left\{ \sum_{r=1}^{q} \zeta \left( s+j+1, \frac{2\xi + 2r - 1}{2q} \right) \exp \left[ \left( \frac{s+j}{2} - \frac{(2\xi + 2r - 1)p}{q} \right) \pi i \right] \right. \]

\[ - \sum_{r=1}^{q} \zeta \left( s+j+1, \frac{2r - 2\xi + 1}{2q} \right) \exp \left[ \left( -\frac{s+j}{2} + \frac{(2r - 2\xi + 1)p}{q} \right) \pi i \right] \}, \]

where \( \ell \in \mathbb{N} \). Further taking \( s = n \) in (3.10) and noting that (2.8) of Theorem 2.1, of course, with

\[ \lambda = e^{2\pi i \xi} \quad \text{and} \quad a = \frac{p}{q} \quad (p \in \mathbb{Z}; \ q \in \mathbb{N}; \ \xi \in \mathbb{R}). \]

Therefore, the formula (3.8) follows. This proof is complete. □

# 4. FURTHER OBSERVATIONS AND CONSEQUENCES

Recently, Srivastava found the following elegant formula for Apostol-Bernoulli polynomials \( B_n(x; \lambda) \) (see [14, p. 84, Eq. (4.6)]):

\[ B_n \left( \frac{p}{q}; e^{2\pi i \xi} \right) = -\frac{n!}{(2q\pi)^n} \left\{ \sum_{j=1}^{q} \zeta \left( n, \frac{\xi + j - 1}{q} \right) \exp \left[ \left( \frac{n}{2} - \frac{2(\xi + j - 1)p}{q} \right) \pi i \right] \right. \]

\[ + \sum_{j=1}^{q} \zeta \left( n, \frac{j - \xi}{q} \right) \exp \left[ \left( -\frac{n}{2} + \frac{2(j - \xi)p}{q} \right) \pi i \right] \} \]

\((n \in \mathbb{N} \setminus \{1\}; \ p \in \mathbb{Z}; \ q \in \mathbb{N}; \ \xi \in \mathbb{R}). \)

When \( \xi \in \mathbb{Z} \) in (4.1), we can deduce a known result given earlier by Cvijović and Klinowski [4, p. 1529, Theorem A]:

\[ B_n \left( \frac{p}{q} \right) = -\frac{2 \cdot n!}{(2q\pi)^n} \sum_{j=1}^{q} \zeta \left( n, \frac{j}{q} \right) \cos \left( \frac{2jp\pi}{q} - \frac{n\pi}{2} \right) \]

\((n \in \mathbb{N} \setminus \{1\}; \ p \in \mathbb{Z}; \ q \in \mathbb{N}). \)
It follows that we set $\alpha = 1$ in (3.4), or $\ell = 1$ in (3.7) and (3.8). Then we obtain the following interesting formula for the Apostol-Euler polynomials $E_n(x; \lambda)$.

**Theorem 4.1** For $n, q \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{R}$, the following formula of the Apostol-Euler polynomials at rational arguments

\[
E_n \left( \frac{p}{q}; e^{2\pi i \xi} \right) = \frac{2 \cdot n!}{(2q\pi)^{n+1}} \times \left\{ \sum_{j=1}^{q} \zeta \left( n + 1, \frac{2\xi + 2j - 1}{2q} \right) \exp \left[ \left( n + 1 \frac{1}{2} - \frac{(2\xi + 2j - 1)p}{q} \right) \pi i \right] 
+ \sum_{j=1}^{q} \zeta \left( n + 1, \frac{2j - 2\xi - 1}{2q} \right) \exp \left[ \left( n + 1 \frac{1}{2} + \frac{(2j - 2\xi - 1)p}{q} \right) \pi i \right] \right\}
\]

holds true in terms of the Hurwitz Zeta function.

A special case of formula (4.3) when $\xi \in \mathbb{Z}$, is just a known result given earlier by Cvijović and Klinowski:

**Corollary 4.2** ([4, p. 1529, Theorem B]) For $n, q \in \mathbb{N}$; $p \in \mathbb{Z}$, the following formula of the classical Euler polynomials

\[
E_n \left( \frac{p}{q} \right) = \frac{4 \cdot n!}{(2q\pi)^{n+1}} \sum_{j=1}^{q} \zeta \left( n + 1, \frac{2j - 1}{2q} \right) \sin \left( \frac{(2j - 1)p\pi}{q} - \frac{n\pi}{2} \right),
\]

holds true in terms of the Hurwitz Zeta function.

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