CONVERGENCE PROPERTIES OF THE $q$-DEFORMED BINOMIAL DISTRIBUTION

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We consider the $q$-deformed binomial distribution introduced by S. C. Jing:


1. INTRODUCTION

The $q$-deformed binomial distribution $QD(n, \tau, q)$ was introduced by Jing [10] in connection with the $q$-deformed boson oscillator and by Chung et al. [5]. Its probabilities are given by

$$
P(X_{QD} = x) = \sum_{x}^{n} \tau^x(\tau; q)_{n-x}, \quad 0 \leq x \leq n, \quad 0 \leq \tau \leq 1, \quad 0 < q < 1,
$$

where

$$
\begin{align*}
\binom{n}{x}_q &= \frac{(q; q)_n}{(q; q)_x(q; q)_{n-x}} \\
(z; q)_n &= \prod_{i=0}^{n-1} (1 - zq^i)
\end{align*}
$$

are the $q$-binomial coefficient and the $q$-shifted Pochhammer symbol; an introduction to the $q$-calculus and basic hypergeometric series can be found in Gasper and Rahman [6]. This distribution was studied by many authors and has applications in physics as well as in approximation theory due to the $q$-Bernstein polynomials and the $q$-Bernstein operator (see Section 2 for details).

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It is well known that for \( n \to \infty \) (and fixed \( \tau \)) the \( q \)-deformed binomial distribution converges to an Euler distribution. This paper is devoted to the study of sequences of \( q \)-deformed binomially distributed random variables \( X_n \sim QD(n, \tau_n, q) \) with parameter sequence \( (\tau_n) \) depending on \( n \) (a similar analysis for Kemp's \( q \)-binomial distribution has been done by Gerhold and Zeiner [7]).

The present paper is organised as follows. In Section 2 we give all definitions of \( q \)-calculus and \( q \)-distributions we need in the following and we sum up some important properties of the \( q \)-deformed binomial distribution. Section 3 deals with parameter sequences \( \tau_n \) where \( \tau_n \) tends to a limit \( c \in [0, 1) \), in particular with the case of constant mean. The pertinent limit law in this case is the Heine distribution and we establish a \( q \)-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution. In Section 4 we investigate parameter sequences with limit 1. Depending on the growth rate of the parameter sequence we obtain a degenerate, a truncated-exponential like or an exponential limit law. Remarkably all these limits are independent of \( q \).

2. NOTATION AND DEFINITIONS

Throughout the paper we use the notation of Gasper and Rahman [6]. Besides the definitions of the \( q \)-binomial coefficient and the \( q \)-shifted Pochhammer symbol we need the \( q \)-number \([x]_q\) of \( x \) defined by

\[
[x]_q := \frac{1-q^x}{1-q};
\]

for \( q \to 1 \) we have \([x]_q \to x \). Moreover, we will need two \( q \)-analogues of the exponential function:

\[
e_q(z) = \frac{1}{(z; q)_\infty}, \quad z \in \mathbb{C} \setminus \{q^{-i} : i = 0, 1, 2, \ldots \}, \quad |q| < 1,
\]

and \( E_q(z) = (-z; q)_\infty \). Here the limit relations \( e_q((1-q)z) \to e^z \) and \( E_q((1-q)z) \to e^z \) hold, as \( q \to 1 \).

The Euler distribution \( E(\lambda, q) \) with parameter \( \lambda \) is defined by

\[
\mathbb{P}(X_E = x) = \frac{\lambda^x}{(q; q)_x} (\lambda; q)_\infty = \frac{\lambda^x}{(q; q)_x} E_q(-\lambda).
\]

This is a \( q \)-analogue of the Poisson distribution since \( E((1-q)\lambda, q) \to P(\lambda) \) for \( q \to 1 \). For properties and applications of this distribution we refer to Johnson, Kemp and Kotz [12], Benkherouf and Bather [1], Biedenharn [2], Kemp [13, 14, 16], Charalambides and Papadatos [4] and Ostrovskaa [19, 20].

Our main object of interest is the \( q \)-deformed binomial distribution \( QD(n, \tau, q) \) defined in (1). This distribution is a \( q \)-analogue of the classical binomial distribution, since in the limit \( q \to 1 \) the \( q \)-deformed binomial distribution with parameter
$(n, \tau, q)$ reduces to the binomial distribution with parameters $(n, \tau)$. The limit $n \to \infty$ of random variables $X_n \sim QD(n, \tau, q)$ leads to an Euler distribution with parameter $\lambda = \tau$. If we denote the probabilities (1) by $p_n(x, \tau)$, then the following recurrence relation holds (see Videnskii [21, Section 3]):

$$p_n(x, \tau) = \tau p_{n-1}(x - 1, \tau) + (1 - \tau)p_{n-1}(x, q\tau).$$

For details and further properties we refer to Jing [10], Jing and Fan [11], Kemp [15, 16], the encyclopedic book Johnson, Kemp and Kotz [12], and to Charalambides [3]. Chung et al. [5], Kupershmidt [17] and Il’inski [8] gave representations of the $q$-deformed binomial distribution as a sum of dependent and not identically distributed random variables.

As mentioned above the $q$-deformed binomial distribution and the Euler distribution appear in particular both in physics ([2, 5, 10, 11]) and in approximation theory. The $q$-Bernstein polynomials of order $n$ are defined by

$$B_n(f(t), q; x) = \sum_{r=0}^{n} f\left( \left[ \frac{[r]}{[n]} \right] \frac{n!}{r!} \right) x^r (x; q)_{n-r},$$

where $f$ is a continuous function on the interval $[0, 1]$. There exists a vast literature on these polynomials, closely related to the distributions under consideration are e.g. [3, 9, 18, 19, 20, 21].

3. PARAMETER SEQUENCES WITH LIMIT $< 1$

In the present section we study sequences of random variables $X_n$ which are $QD(n, \tau_n, q)$-distributed, where the parameters $\tau_n$ converge to a limit $c \in [0, 1)$. In particular we prove a $q$-analogue of the convergence of the classical binomial distribution with constant mean to a Poisson distribution.

As noted above the sequence converges in the case of constant parameters $\tau_n = \tau$ to an Euler distribution with parameter $\tau$. The following proposition is a mild generalisation of the convergence to an Euler distribution mentioned in the previous section and shows that the Euler distribution is the limit distribution for every convergent parameter sequence $\tau_n$ with limit in $[0, 1)$.

**Proposition 3.1.** Let $X_n \sim QD(n, \tau_n, q)$. Then, for $n \to \infty$,

$$X_n \to E(\tau, q)$$

if $\tau_n \to \tau$ and $0 \leq \tau < 1$.

**Proof.** Note that

$$\mathbb{P}(X_n = x) = \left[ \frac{n}{x} \right] \tau_n^x \prod_{i=0}^{n-x} (1 - \tau_n q^i).$$
Therefore exists an \( n \) such that the limit is commutative: \( X \lim_{n \to \infty} X \) converges to a Poisson distribution for \( q \to 1 \). From the previous theorem we deduce immediately the following corollary.

**Corollary 3.2.** Let \( X_n \sim QD(n, \tau_n(q), q) \) with \( \tau_n(q) \to \tau(q) \) for \( n \to \infty \) with the additional property \( \tau(q) \to \lambda \) in the limit \( q \to 1 \) (recall that we assume \( \tau(q) < 1 \) in this section). Then the following diagram is commutative:

\[
\begin{array}{ccc}
QD(n, \tau_n, q) & \xrightarrow{n \to \infty} & E(\tau(q), q) \\
q^{-1} & \downarrow & q^{-1} \\
B \left( n, \frac{\lambda}{n} \right) & \xrightarrow{n \to \infty} & P(\lambda)
\end{array}
\]

One very natural way to choose the parameters is to set \( \tau_n = \frac{\lambda}{[n]_q} \).

Our next goal is to establish a convergence result, which is analogous to the convergence of the classical binomial distribution with constant mean to a Poisson distribution and reduces in the limit \( q \to 1 \) to that theorem. For this purpose we start with an elementary fact.

**Lemma 3.3.** Let \( f_n(x) \), \( n \in \mathbb{N} \), be a sequence of continuous functions which converges pointwise to a continuous limit \( f(x) \). Assume that for each \( n \) the function \( f_n(x) \) has a single root \( \hat{x}_n \), and \( f(x) \) has a single root \( \hat{x} \), and that \( f(y)f(z) < 0 \) for \( y < \hat{x} \) and \( z > \hat{x} \). Then \( \hat{x}_n \to \hat{x} \).

**Proof.** W.l.o.g. we may assume that \( f(z) > 0 \) for \( z > \hat{x} \). For given \( \varepsilon > 0 \) choose a \( \delta(\varepsilon) < \min(f(\hat{x} + \varepsilon), f(\hat{x} - \varepsilon)) \). Then there exists an \( N = N(\delta(\varepsilon)) \) such that for all \( n \geq N \) we have \( |f_n(\hat{x} + \varepsilon) - f(\hat{x} + \varepsilon)| < \delta(\varepsilon) \). Therefore \( f_n(\hat{x} + \varepsilon) > 0 \). Moreover there exists an \( M = M(\delta(\varepsilon)) \) such that for all \( n \geq M \) we have \( |f_n(\hat{x} - \varepsilon) - f(\hat{x} - \varepsilon)| \leq \delta(\varepsilon) \). Therefore \( f_n(\hat{x} - \varepsilon) < 0 \). Hence, by continuity, for all \( n \geq \max(N, M) \) we have \( |\hat{x} - \hat{x}_n| < 2\varepsilon \).

The essential key to apply this lemma is the following representation of the means \( \mu_n(\tau, q) \), which allows us to extract important properties of the means easily.

**Lemma 3.4.** The means \( \mu_n(\tau, q) \) have the representation

\[
\mu_n(\tau, q) = \sum_{j=1}^{n} \binom{n}{j} \tau^j.
\]

**Proof.** We proceed by induction. For \( n = 1 \) this is obviously true. Now suppose that the statement is true for \( n - 1 \). In order to calculate \( \mu_n(\tau, q) \) we use the
recurrence relation (2). Hence we have

\[ \mu_n(\tau, q) = \sum_{x=1}^{n} xp_n(x, \tau) = \tau \sum_{x=1}^{n} xp_{n-1}(x-1, \tau) + (1 - \tau) \sum_{x=1}^{n-1} xp_{n-1}(x, q\tau). \]

Shifting the summation index in the first sum, splitting this sum and using the induction hypothesis yields

\[ \mu_n(\tau, q) = \tau \sum_{j=1}^{n-1} (q q_j^{j-1}) \left[ \begin{array}{c} n-1 \\ j_q \end{array} \right] \tau^j + \sum_{x=1}^{n} \tau p_{n-1}(x-1, \tau) \\
+ (1 - \tau) \sum_{j=1}^{n-1} (q q_j^{j-1}) \left[ \begin{array}{c} n-1 \\ j_q \end{array} \right] \tau^j q^j. \]

The second sum reduces to \( \tau \). Collecting powers of \( \tau \) gives

\[ \mu_n(\tau, q) = \tau \left( 1 + \left[ \begin{array}{c} n-1 \\ 1_q \end{array} \right] \right) \\
+ \sum_{j=2}^{n} (q q_j^{j-1}) \left[ \begin{array}{c} n-1 \\ j_q \end{array} \right] + (q q_j^{j-2}) \left[ \begin{array}{c} n-1 \\ j-1_q \end{array} \right] (1 - q^{j-1}) \tau^j. \]

Consequently the desired result follows by the recurrence relation for the \( q \)-binomial coefficients (see e.g. [6, (1.45)]).

**Remark 3.5.** An alternative way to prove this lemma is to use Kemp’s [15, p. 300] representation of the probability generating function, to differentiate and to manipulate the sum.

Using the monotonicity of the \( q \)-binomial coefficients in \( n \) we immediately get the following proposition.

**Proposition 3.6.** The means \( \mu_n(\tau, q) \) are strictly increasing in \( n \) (for \( \tau > 0 \)) and \( \tau \).

Now we turn to the convergence result:

**Theorem 3.7.** Fix \( \mu > 0 \) and choose the parameter \( \tau_n = \tau_n(q, \mu) \) of the \( q \)-deformed binomial distribution such that \( \mu_n = \mu \). Then we have

(i) The sequence \( QD(n, \tau_n, q) \) converges for \( n \to \infty \) to an Euler distribution \( E(\tau, q) \), where \( \tau = \lim_{n \to \infty} \tau_n \).

(ii) For fixed \( n \), \( QD(n, \tau_n, q) \) tends to a binomial distribution \( B\left(n, \frac{\mu}{n}\right) \) in the limit \( q \to 1 \).

(iii) For \( q \to 1 \), the Euler distribution \( E(\tau, q) \) converges to a Poisson distribution with parameter \( \mu \).
So we obtain the following commutative diagram:

\[
\begin{array}{ccc}
QD(n, \tau_n(q), q) & \xrightarrow{n \to \infty} & E(\tau(q), q) \\
q \downarrow & & \downarrow \scriptstyle{q \to 1} \\
B\left(n, \frac{\mu}{n}\right) & \xrightarrow{n \to \infty} & P(\mu)
\end{array}
\]

**Proof.** First we check that for given \(\mu\), \(q\) and large \(n\) there exists a unique \(\tau_n\) with \(\mu_n(\tau_n, q) = \mu\). The function \(\mu_n(\tau, q)\) is continuous and strictly increasing in \(n\) and \(\tau\) by the previous theorem. Moreover, we have \(\lim_{\tau \to 0} \mu_n(\tau, q) = 0\). If we choose \(\tau_n\) such that \(\tau_n \to 1\) then \(\mu_n(\tau_n, q)\) becomes arbitrarily large. Consequently there is a unique solution of \(\mu_n(\tau, q) = \mu\). By Lemma 3.3 the sequence \(\tau_n\) converges to a limit \(\tau\) where \(\tau\) is the unique solution of \(\mu_E(\tau, q) = \mu\), where \(\mu_E(\tau, q)\) is the mean of an Euler-distribution with parameters \(\tau\) and \(q\). This mean can be written as

\[
\mu_E(\tau, q) = \sum_{i=0}^{\infty} \frac{q^i \tau}{1 - q^i \tau},
\]

see [13] or take the limit \(n \to \infty\) (using the dominated convergence theorem) in Lemma 3.4 and manipulate the sum (i.e. expand the denominator as a geometric series and change the order of summation).

Again by Lemma 3.3 we get that \(\tau_n \to \mu/n\). It remains to check that \(\tau/(1-q)\) converges to \(\mu\) in the limit \(q \to 1\). But this is again a consequence of Lemma 3.3 since \(\tau/(1-q)\) is the unique solution of \(\mu_E((1-q)\tau, q) = \mu\) and \(\mu_E((1-q)\tau, q)\) tends to \(\tau\) for \(q \to 1\). \(\square\)

### 4. Parameter Sequences with Limit 1

In this section we investigate sequences \(X_n\) of random variables, where \(X_n\) is \(QD(n, \tau_n, q)\)-distributed and the parameters \(\tau_n\) converge to 1. The behaviour of the sequences \(X_n\) depends on the growth rate of \(\tau_n\). Therefore we will distinguish three cases: Firstly we examine the case \(\tau_n^\mu \to 1\), where it will turn out that the limit distribution is degenerate. Then we study the case \(\tau_n^\mu \to c\) with \(0 < c < 1\). Here the limit law depends only on \(c\) and is a truncated exponential distribution. Finally we turn to the case \(\tau_n^{f(n)} \to c\) where \(0 < c < 1\) and \(f(n) = o(n)\); this will lead to an exponential distribution.

Consider sequences of random variables \(X_n \sim QD(n, \tau_n, q)\) with \(\tau_n \to 1\) and additionally \(\tau_n^\mu \to 1\) first. Then we have the following theorem:

**Theorem 4.1.** Let \(X_n \sim QD(n, \tau_n, q)\) with \(\tau_n \to 1\) and \(\tau_n^\mu \to 1\). Then \(n - X_n\) converges to the point measure at \(0\).

**Proof.** The probability that \(Y_n = n - X_n\) is equal to 0 is given by

\[
P(Y_n = 0) = \tau_n^\mu\]
which converges to 1 by assumption.

Now let us investigate sequences $X_n \sim QD(n, \tau_n, q)$, where $\tau_n \to 1$ and $\tau_n^n \to c$ for a $c \in (0, 1)$. Before we can establish the distribution of the limit of such a sequence, we start with several lemmas, which allow us to compute the asymptotic behaviour of certain sums of probabilities of $QD(n, \tau_n, q)$-distributed random variables and their means and variances.

The first lemma is an analogue to Lemma 3.4 and gives an alternative representation of the variance:

**Lemma 4.2.** The second moment of $X_n(\tau, q)$ can be written as

$$
\sum_{x=1}^{n} x^2 \left[ \begin{array}{c} n \\ x \\ q \end{array} \right] \tau^x(\tau; q)_{n-x} = \sum_{j=1}^{n} n a_j \tau^j
$$

with

$$
n a_j = \left[ \begin{array}{c} n \\ j \\ q \end{array} \right] (q; q)_{j-1} \left( 1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^i} \right).
$$

**Proof.** We prove this by induction. The case $n = 1$ is obvious. To compute $E(X_n^2)$ we use the recurrence (2) again and shift the summation index. This gives

$$
V_n := \sum_{x=1}^{n} x^2 p_n(x, \tau) = \tau \sum_{x=0}^{n-1} (x^2 + 2x + 1)p_{n-1}(x, \tau) + (1 - \tau) \sum_{x=1}^{n} x^2 p_{n-1}(x, q\tau).
$$

By splitting sums and by using Lemma 3.4 and the induction hypothesis we find

$$
V_n = \tau \sum_{j=1}^{n-1} n-1 a_j \tau^j + 2\tau \sum_{j=1}^{n-1} (q; q)_{j-1} \left[ \begin{array}{c} n-1 \\ j \\ q \end{array} \right] \tau^j + \tau + (1 - \tau) \sum_{j=1}^{n-1} n-1 a_j q^j \tau^j.
$$

Collecting powers of $\tau$ yields

$$
V_n = \tau \left( 1 + \left[ \begin{array}{c} n-1 \\ 1 \\ q \end{array} \right] \right)
+ \sum_{j=2}^{n} \left( n-1 a_{j-1} (1 - q^{j-1}) + 2 \left[ \begin{array}{c} n-1 \\ j-1 \\ q \end{array} \right] (q; q)_{j-2} + n-1 a_j q^j \right) \tau^j.
$$

The first term gives $\left[ \begin{array}{c} n \\ 1 \\ q \end{array} \right] \tau$ and the coefficient of $\tau^j$ in the sum equals

$$
\left[ \begin{array}{c} n-1 \\ j-1 \\ q \end{array} \right] (q; q)_{j-2} \left( 1 + 2 \sum_{i=1}^{j-2} \frac{1}{1 - q^i} \right) \left[ 1 - q^{j-1} \right] + 2 \left[ \begin{array}{c} n-1 \\ j-1 \\ q \end{array} \right] (q; q)_{j-2}
+ \left[ \begin{array}{c} n-1 \\ j \\ q \end{array} \right] (q; q)_{j-1} \left( 1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^i} \right) q^j.
$$
which implies the statement by using the recurrence relation of the $q$-binomial coefficients again.

The next three lemmas are devoted to the asymptotic behaviour of sums of powers of $\theta_n$, where $0 < \theta_n < 1$ and $\theta_n \to 1$.

**Lemma 4.3.** If $f(n) \to \infty$ for $n \to \infty$ and $\theta_n \leq 1$ such that $\theta_n^{f(n)} \to c$ with $0 < c < 1$, then

$$\sum_{i=0}^{\infty} \theta_n^i \sim -\frac{f(n)}{\log c}, \quad n \to \infty.$$  

**Proof.** Since $c < 1$ almost all $\theta_n$ must be smaller than 1. Thus we assume w.l.o.g. that $\theta_n < 1$ and obtain

$$\sum_{i=0}^{\infty} \theta_n^i = \frac{1}{1-\theta_n} \sim -\frac{1}{\log \theta_n}$$

using the substitution $\theta_n = 1 + x_n$ in the elementary equivalence

$$\log(1 + x) \sim x, \quad x \to 0.$$  

Since $f(n) \log \theta_n \sim \log c$, the statement follows. □

**Lemma 4.4.** For $\theta_n \leq 1$ and $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $c \in (0,1)$ and $g(n)/f(n) \sim \beta$, $g(n) \leq n$ we have

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i \sim c^\beta - \frac{1}{\log c} f(n)$$

and

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \cr i \end{bmatrix}_q \theta_n^i \sim c_q^\beta - \frac{1}{\log c} f(n)$$

as $n \to \infty$.

**Proof.** We rewrite the first sum as

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i = \frac{1 - g_n^{\lfloor g(n) \rfloor} + 1}{1 - \theta_n}.$$  

The growth of the denominator is given in Lemma 4.3, and the numerator tends to $1 - c^\beta$, since $\theta_n^{g(n)} = \theta_n^{\lfloor g(n) \rfloor} \to c^\beta$ because of $\theta_n \to 1$.

To get the asymptotic of the second sum we write

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \cr i \end{bmatrix}_q \theta_n^i = \sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \cr i \end{bmatrix}_q \theta_n^i + \sum_{i=0}^{\lfloor g(n) \rfloor - 1} \begin{bmatrix} n \cr i \end{bmatrix}_q \theta_n^i + \sum_{i=0}^{\lfloor g(n) \rfloor - \lfloor g(n) \rfloor} \begin{bmatrix} n \cr i \end{bmatrix}_q \theta_n^i.$$
The first and the third sum on the right-hand side are $\mathcal{O}(\sqrt{g(n)})$ and therefore asymptotically negligible. The second sum is bounded by

$$\frac{(q;q)_n}{(q;q)_{\lfloor \sqrt{g(n)} \rfloor + 1}(q;q)_{n - \lfloor \sqrt{g(n)} \rfloor}} \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \frac{n^i}{\theta_n^i} \leq \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \frac{n_i}{\theta_n^i} \leq \frac{(q;q)_n}{(q;q)_{\lfloor n/2 \rfloor}} \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \theta_n^i.$$ 

By the first part of this lemma the lower and the upper bound has the asserted asymptotic. \hfill $\square$

**Lemma 4.5.** If $\theta_n \leq 1$ and $\theta_n \to 1$ with $\theta_n^c \to c$ for $0 < c < 1$, then

$$\sum_{i=0}^{n} i \theta_n^i \sim \frac{1 - c + c \log c}{\log^2 c} n^2$$

and

$$\sum_{i=0}^{n} \frac{n_i}{\theta_n^i} \sim \frac{1 - c + c \log c}{\log^2 c} n^2$$

as $n \to \infty$.

**Proof.** To estimate this sum we use Lemma 4.3 again and the identity

$$\sum_{i=0}^{n} i t^i = \frac{t(1 - t^n - n t^n (1 - t))}{(1 - t)^2}.$$ 

Hence, setting $t = \theta_n$,

$$\sum_{i=0}^{n} i \theta_n^i \sim (1 - c - n \theta_n^c (1 - \theta_n)) \frac{n^2}{\log^2 c} \sim (1 - c + c \log c) \frac{n^2}{\log^2 c}.$$ 

Here we used that under the assumption $\theta_n^c \to c$ we have $(1 - \theta_n)n \to -\log c$. This can easily be seen from the equivalence (3). The asymptotic for the sum with the $q$-binomial coefficient is obtained as in Lemma 4.4. \hfill $\square$

Now we are ready to establish the essential key in proving the convergence result: we give the asymptotic behaviour of sums of probabilities and the means and variances of $QD(n, \tau_n, q)$-distributed random variables.

**Lemma 4.6.** Let $X_n$ be $QD(n, \tau_n, q)$-distributed and denote by $\mu_n(\tau_n, q)$ and $\sigma_n^2(\tau_n, q)$ the corresponding mean and variance. If $\tau_n \to 1$ and $\tau_n^c \to c$ with
0 < c < 1 and \( f(n) \sim \beta n, \ f(n) < n \), then

\[
\sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} \sim 1 - c^\beta,
\]

\[
\mu_n(\tau_n, q) \sim \frac{c - 1}{\log c} n,
\]

\[
\sigma_n^2(\tau_n, q) \sim \frac{1 + 2c \log c - c^2}{(\log c)^2} n^2,
\]

as \( n \to \infty \).

**Proof.** We start with the first assertion. Since \( f(n) < n \) we can write

\[
S_n := \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} = (1 - \tau_n) \sum_{x=0}^{\lfloor f(n) \rfloor} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} \prod_{i=1}^{n-x} (1 - \tau_n q^i).
\]

The summands are bounded by \( e_q(q)^2 \), hence

\[
S_n \sim (1 - \tau_n) \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} \prod_{i=1}^{n-x} (1 - \tau_n q^i) =: \hat{S}_n.
\]

Estimating the product and using again the boundedness of the summands yields

\[
\hat{S}_n \leq (1 - \tau_n)(\tau_n; q)_{n - [f(n)] - [\sqrt{n}]} \sum_{x=\lfloor \sqrt{n} \rfloor}^{\lfloor f(n) \rfloor} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} \sim (1 - \tau_n)(\tau_n; q)_{n - [f(n)] - [\sqrt{n}]} \sum_{x=\lfloor \sqrt{n} \rfloor}^{[f(n)] - [\sqrt{n}]} \tau_n^x \left( \frac{n}{x} q \right) )_{n-x} =: \hat{\hat{S}}_n.
\]

As in the proof of Proposition 3.1 and with use of Lemma 4.4 (with \( g(n) := f(n) \) and \( f(n) := n \)) we obtain

\[
\hat{\hat{S}}_n \sim (1 - \tau_n) \frac{1}{e_q(q)} e_q(q) e^\beta - \frac{1}{\log c} n \sim 1 - c^\beta.
\]

In an analogous way we find a lower bound of \( \hat{S}_n \) that is asymptotically equivalent to \( 1 - c^\beta \).

Now we prove the second proposition of the lemma: Use Lemma 3.4, easy estimates of the \( q \)-Pochhammer symbol and the asymptotics given in Lemma 4.4 to obtain

\[
\mu_n(\tau_n, q) \leq \sum_{j=1}^{[\sqrt{n}]} \frac{(q; q)_n}{(q; q)_{[n/2]}} + (q; q)_{[\sqrt{n}]} \sum_{j=[\sqrt{n}]}^{n} \tau_n^j \sim \frac{1}{e_q(q)} e_q(q) e^\beta - \frac{1}{\log c} n
\]
and
\[ \mu_n(\tau_n, q) \geq \langle q; q \rangle_n \sum_{j=1}^{n} \left[ \frac{n}{j} \right] \tau_n^{j} \sim \frac{c - 1}{\log c} n. \]

Similarly we proceed for the second moments of \( X_n(\tau_n, q) \) and estimate with use of Lemma 4.5
\[
E(X_n^2) \geq \sum_{j=1}^{n} \langle q; q \rangle_{j-1} (1 + 2(j - 1)) \left[ \frac{n}{j} \right] \tau_n^{j} \\
\geq 2\langle q; q \rangle_n \sum_{j=1}^{n} (j - 1) \left[ \frac{n}{j} \right] \tau_n^{j} \sim 2 \frac{1 - c + c \log c}{(\log c)^2} n^2.
\]

To bound the second moment from above we split the sum into two parts
\[
E(X_n^2) \leq \sum_{j=1}^{[\sqrt{n}]} \frac{\langle q; q \rangle_n}{\langle q; q \rangle_{[\sqrt{n}]}} \left( 1 + \frac{2n}{1 - q} \right) \\
+ \sum_{j=[\sqrt{n}]}^{n} \langle q; q \rangle_{j-1} \left( 1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^i} \right) \left[ \frac{n}{j} \right] \tau_n^{j} \\
+ \sum_{j=[\sqrt{n}]}^{n} \langle q; q \rangle_{j-1} \left( 1 + 2 \sum_{i=1}^{[\sqrt{j}]} \frac{1}{1 - q^i} \right) \left[ \frac{n}{j} \right] \tau_n^{j}.
\]

The first sum is \( o(n^2) \), and splitting the inner sum in the second term we obtain
\[
E(X_n^2) = o(n^2) + \sum_{j=[\sqrt{n}]}^{n} \langle q; q \rangle_{j-1} \left( 1 + 2 \sum_{i=[\sqrt{j}]}^{j-1} \frac{1}{1 - q^i} \right) \left[ \frac{n}{j} \right] \tau_n^{j} \\
+ \sum_{j=[\sqrt{n}]}^{n} \langle q; q \rangle_{j-1} \left( 1 + 2 \sum_{i=1}^{[\sqrt{j}]} \frac{1}{1 - q^i} \right) \left[ \frac{n}{j} \right] \tau_n^{j}.
\]

Here the first sum is \( o(n^2) \) again and easy estimates of the second term yield
\[
E(X_n^2) \leq o(n^2) + 2\langle q; q \rangle_{[\sqrt{n}]} \sum_{j=[\sqrt{n}]}^{n} j \frac{1}{1 - q^{j-[\sqrt{j]}-1}} \left[ \frac{n}{j} \right] \tau_n^{j} \\
\leq o(n^2) + 2\langle q; q \rangle_{[\sqrt{n}]} \frac{1}{1 - q^{n-[\sqrt{n]}-1}} \sum_{j=1}^{n} \left[ \frac{n}{j} \right] \tau_n^{j} \\
\sim 2 \frac{1 - c + c \log c}{(\log c)^2} n^2.
\]

Thus
\[
E(X_n^2(\tau_n, q)) \sim 2 \frac{1 - c + c \log c}{(\log c)^2} n^2.
\]
Hence
\[ \sigma_n^2(\tau_n, q) = E(X_n^2(\tau_n, q)) - \mu_n(\tau, q)^2 \sim \left( \frac{2}{2} - c + c \log c \right) \frac{(c - 1)^2}{(\log c)^2} n^2 \]
\[ \sim \frac{1 + 2c \log c - c^2}{(\log c)^2} n^2, \]
which completes the proof.

After this analysis of the means and variances it is now easy to obtain the limiting distribution of the sequence \( X_n \).

**Theorem 4.7.** Let \( Y_n \sim QD(n, q, \tau_n) \) with \( \tau_n \to 1 \) and \( \tau_n^2 \to c \) with \( 0 < c < 1 \). Then the sequence of the normalised random variables \( X_n = (Y_n - \mu_n) / \sigma_n \) converges to a limit \( X \) with
\[ P(X \leq x) = 1 - e^{-1} \frac{c - 1}{\sqrt{1 + 2c \log c - c^2}} x \]
for
\[ x \in \left[ -\frac{1 - c}{\sqrt{1 + 2c \log c - c^2}} - \frac{c - \log c - 1}{\sqrt{1 + 2c \log c - c^2}} \right] \]
and
\[ P(X \leq x) = 1 \quad \text{for} \quad x = \frac{c - \log c - 1}{\sqrt{1 + 2c \log c - c^2}}. \]

**Proof.** The support of \( X \) is given by
\[ \left[ \lim_{n \to \infty} \frac{\mu_n(\tau_n, q)}{\sigma_n(\tau_n, q)}, \lim_{n \to \infty} \frac{n - \mu_n(\tau_n, q)}{\sigma_n(\tau_n, q)} \right]. \]
Using Lemma 4.6 the stated support follows immediately.

Computing the distribution function of \( X \) yields with use of Lemma 4.6
\[ P(X_n \leq x) = \sum_{0 \leq y \leq \sigma_n x + \mu_n} \tau_n^y \binom{n}{y}_q (\tau_n; q)_{n-y} \sim 1 - e^x \]
with
\[ x = \frac{c - \log c - 1}{\sqrt{1 + 2c \log c - c^2}} \]
for
\[ x < \frac{c - \log c - 1}{\sqrt{1 + 2c \log c - c^2}}. \]
Simplifying \( e^x \) yields the theorem.

Now we turn to the third case, which treats sequences of random variables \( X_n \sim QD(n, \tau_n, q) \) where \( \tau_n \to 1 \) and \( \tau_n^f(n) \to c \) for a \( c \in (0, 1) \) and \( f(n) = o(n) \).
This case is very similar to the previous one, and so we start with an analogue of Lemma 4.5.

**Lemma 4.8.** Let \( f(n) \to \infty, f(n) = o(n), \theta_n^{f(n)} \to c \) with \( 0 < c < 1 \). Then

\[
\sum_{i=0}^{n} i \theta_n^i \sim \frac{f(n)^2}{\log^2 c} \quad \text{and} \quad \sum_{i=0}^{n} \left[ \frac{n}{i} \right] \theta_n^i \sim e_q(q) \frac{f(n)^2}{\log^2 c}
\]
as \( n \to \infty \).

**Proof.** Follow the proof of Lemma 4.5 and observe that \( n \theta_n^i (1 - \theta_n) \) tends to zero.

Following the proof of Lemma 4.6 and using Lemma 4.8 instead of Lemma 4.5 we obtain

**Lemma 4.9.** If \( \tau_n \to 1 \) and \( \tau_n^{f(n)} \to c \) with \( 0 < c < 1 \) and \( f(n) = o(n), g(n) \sim \beta f(n) \), then

\[
\sum_{x=0}^{\lfloor g(n) \rfloor} \tau_n^{x \left\lfloor \frac{n}{q} \right\rfloor} (\tau_n; q)_{n-x} \sim 1 - e^\beta,
\]

\[
\mu_n(\tau_n, q) \sim \frac{-f(n)}{\log c},
\]

\[
\sigma_n^2(\tau_n, q) \sim \frac{f(n)^2}{(\log c)^2},
\]
as \( n \to \infty \).

As an immediate consequence we get the distribution of the limit of \( X_n \), which is an exponential distribution and is again independent of \( q \).

**Theorem 4.10.** Let \( Y_n \sim QD(n, q, \tau_n) \) with \( \tau_n \to 1 \) and \( \tau_n^{f(n)} \to c \) with \( 0 < c < 1 \) and \( f(n) = o(n) \). Then the sequence of the normalised random variables \( X_n = (Y_n - \mu_n)/\sigma_n \) converges to a normalised exponential distribution with parameter 1, i.e.

\[
P(X \leq x) = 1 - e^{-x-1}, \quad x \geq -1.
\]

**Proof.** Lemma 4.9 yields immediately that the support of the limit distribution is \([-1, \infty)\). Computing the distribution function gives

\[
P(X \leq x) = \sum_{0 \leq y \leq \tau_n + \mu_n} \tau_n^{y \left\lfloor \frac{n}{q} \right\rfloor} (\tau_n; q)_{n-y} \sim 1 - e^{-\frac{x+1}{\log c}} = 1 - e^{-x-1}.
\]

Comparing this result with Theorem 3.7 we see that this corresponds to taking the limit \( c \to 0 \).

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