ON A BINOMIAL COEFFICIENT AND A PRODUCT OF PRIME NUMBERS

Horst Alzer, József Sándor

Let \( p_n \) be the \( n \)-th prime number. We prove the following double-inequality. For all integers \( k \geq 5 \) we have

\[
\exp[k(c_0 - \log \log k)] \leq \binom{k^2}{k} \leq \exp[k(c_1 - \log \log k)]
\]

with the best possible constants

\[
c_0 = \frac{1}{5} \log 23 + \log \log 5 = 1.10298 \ldots
\]

and

\[
c_1 = \frac{1}{192} \log \left( \frac{36864}{192} \right) + \log \log 192 - \frac{1}{192} \log \left( p_1 \cdot p_2 \cdots p_{192} \right) = 2.04287 \ldots
\]

This refines a result published by Gupta and Khare in 1977.

1. INTRODUCTION

The work on this note has been inspired by a remarkable short paper published by Gupta and Khare [4] in 1977. The authors presented a connection between the binomial coefficient \( \binom{k^2}{k} \) and the product of the first \( k \) prime numbers.

2010 Mathematics Subject Classification. 05A10, 11A41.
Keywords and Phrases. Prime numbers, binomial coefficients, Chebychev’s function, inequalities, monotonicity.
Proposition. If \( k = 3, 4, \ldots, 1793 \), then
\[
\binom{k^2}{k} > p_1 \cdot p_2 \cdots p_k,
\]
whereas, if \( k \geq 1794 \), then
\[
\binom{k^2}{k} < p_1 \cdot p_2 \cdots p_k.
\]

The Proposition implies an interesting number theoretical theorem: if \( 1794 \leq k \leq n \leq k^2 \), then \( \binom{n}{k} \) has less than \( k \) distinct prime divisors. This improves an earlier result of Erdős, Gupta and Khare [3].

From the Proposition we obtain
\[
Q_k = \frac{\binom{k^2}{k}}{p_1 \cdot p_2 \cdots p_k} < 1 \quad \text{for} \quad k \geq 1794.
\]

It is natural to look for a refinement of (1.1). More precisely, we ask for the largest number \( c_0 \) and the smallest number \( c_1 \) such that the double-inequality
\[
(1.2) \quad \exp[k(c_0 - \log \log k)] \leq Q_k \leq \exp[k(c_1 - \log \log k)]
\]
holds for all \( k \geq 5 \). It is our aim to solve this problem. In the next section, we demonstrate that (1.2) is valid with \( c_0 = 1.10298 \ldots \) and \( c_1 = 2.04287 \ldots \). This provides not only a positive lower bound for \( Q_k \), but improves (1.1) for all \( k \geq 2237 \).

2. MAIN RESULT

In order to offer sharp bounds for \( Q_k \) we need the following estimates.

Lemma 1. For \( k \geq 2 \) we have
\[
(2.1) \quad k^2 \log k - \left( k^2 - k + \frac{1}{2} \right) \log(k - 1) - \frac{1}{2} \log(2\pi) - \frac{1}{12k} - \frac{1}{12(k - 1)k^2}
< \log \binom{k^2}{k} < k^2 \log k - \left( k^2 - k + \frac{1}{2} \right) \log(k - 1) - \frac{1}{2} \log(2\pi).
\]

This lemma is due to Sasvári [9]. Moreover, we need upper and lower bounds for the Chebyshev function
\[
\theta(x) = \sum_{p \leq x} \log p,
\]
where \( p \) runs over all prime numbers \( \leq x \).
Lemma 2. We have for \( k \geq 3 \)

\[
(2.2) \quad \frac{1}{k} \theta(p_k) = \frac{1}{k} \sum_{j=1}^{k} \log(p_j) \geq \log k + \log \log k - 1 + \frac{\log \log k - 2.1454}{\log k}
\]

and for \( k \geq 126 \)

\[
(2.3) \quad \frac{1}{k} \theta(p_k) \leq \log k + \log \log k - 1 + \frac{\log \log k - 1.9185}{\log k}.
\]

A proof of Lemma 2 can be found in [7]; see also [2]. We are now in a position to present our main result.

Theorem. For all integers \( k \geq 5 \) we have

\[
(2.4) \quad \exp[k(c_0 - \log \log k)] \leq \frac{k^2}{p_1 \cdot p_2 \cdots p_k} \leq \exp[k(c_1 - \log \log k)]
\]

with the best possible constants

\[
(2.5) \quad c_0 = \frac{1}{5} \log 23 + \log \log 5 = 1.10298 \ldots
\]

and

\[
(2.6) \quad c_1 = \frac{1}{192} \log \left( \frac{36864}{192} \right) + \log \log 192 - \frac{1}{192} \log (p_1 \cdot p_2 \cdots p_{192}) = 2.04287 \ldots.
\]

Proof. A short calculation reveals that (2.4) is equivalent to

\[
c_0 \leq f(k) \leq c_1
\]

with

\[
(2.7) \quad f(k) = \frac{1}{k} \log \left( \frac{k^2}{k} \right) + \log \log k - \frac{1}{k} \theta(p_k).
\]

First, we prove that

\[
(2.8) \quad 1.9114 < f(k) \quad \text{for} \quad k \geq 126.
\]

Let \( k \geq 126 \). The left-hand side of (2.1) and the elementary inequalities

\[
\frac{1}{k} < \log k - \log(k-1) < \frac{1}{k-1}
\]

imply

\[
(2.9) \quad \log \left( \frac{k^2}{k} \right) > k + \left(k - \frac{1}{2}\right) \log(k-1) - \alpha_0 > k + \left(k - \frac{1}{2}\right) \log k - \alpha_1
\]
with
\[ \alpha_0 = \frac{1}{2} \log(2\pi) + \frac{1}{12 \cdot 126} + \frac{1}{12 \cdot 125 \cdot 126 \cdot 126} \]
and
\[ \alpha_1 = \alpha_0 + 1 + \frac{1}{2 \cdot 125} = 1.9235\ldots \]

Using (2.3), (2.7), and (2.9) leads to
\[ f(k) > 2 \left( \frac{\log k}{2k} + \frac{1.9236}{k} + \frac{\log \log k - 1.9185}{\log k} \right). \]

Since
\[ \frac{\log k}{2k} + \frac{1.9236}{k} < 0.0345 \quad \text{and} \quad \frac{\log \log k - 1.9185}{\log k} < 0.0541, \]
we get
\[ f(k) > 2 - 0.0345 - 0.0541 = 1.9114\ldots. \]

This settles (2.8). By direct computation we find
\[ (2.10) \quad \min_{5 \leq k \leq 125} f(k) = f(5) = 1.10298\ldots. \]

From (2.8) and (2.10) we conclude that the first inequality in (2.4) holds for \( k \geq 5 \) with the best possible constant given in (2.5).

Applying the second inequality in (2.1), (2.2), and (2.7) yields for \( k \geq 652 \):
\[ (2.11) \quad f(k) < 1 + (k - 1) \log \left( 1 + \frac{1}{k - 1} \right) - \frac{\log(2\pi(k - 1))}{2k} - \frac{\log \log k - 2.1454}{\log k} < 2 - \frac{\log \log k - 2.1454}{\log k} < 2.0428. \]

Moreover, we have
\[ (2.12) \quad \max_{5 \leq k \leq 651} f(k) = f(192) = 2.04287\ldots. \]

From (2.11) and (2.12) we obtain that the right-hand side of (2.4) is valid with the best possible constant \( c_1 \) given in (2.6). \( \square \)

3. REMARKS

(i) The behaviour of \( f(k) \) for large \( k \) is quite surprising. We have the limit relation
\[ \lim_{k \to \infty} f(k) = 2, \]
although computer calculations show that \( f(k) \) is decreasing in the range \( 1000 \leq k \leq 1500000 \) with \( f(10000) = 1.9908\ldots. \).
(ii) More inequalities involving the product $p_1 \cdot p_2 \cdots p_k$ can be found in [1], [5, p. 246], [6].

(iii) Let $\psi = \Gamma'/\Gamma$ be the logarithmic derivative of Euler’s gamma function. We define for $x \geq 2$:

$$\phi(x) = \log \frac{\Gamma(x^2 + 1)}{\Gamma(x + 1) \Gamma(x^2 - x + 1)}.$$

Differentiation gives

$$\phi'(x) = 2x[\psi(x^2 + 1) - \psi(x^2 - x + 1)] + [\psi(x^2 - x + 1) - \psi(x + 1)].$$

Since $\psi$ is strictly increasing on $(0, \infty)$, we conclude that $\phi'(x)$ is positive for $x \geq 2$. Hence,

$$\phi(k) = \log \binom{k^2}{k} < \log \binom{(k+1)^2}{k+1} = \phi(k+1) \quad \text{for} \quad k \geq 2.$$ 

We have $\phi(1) = 0 < \log 6 = \phi(2)$. Thus we obtain: If $\mu \geq 0$, then the sequence

$$\Delta_k(\mu) = \binom{k^2}{k} \exp(\mu p_k)$$

is strictly increasing for $k \geq 1$. This result leads to the question: Does there exist a negative real number $\mu_0$ and an integer $k_0$ such that $\Delta_k(\mu_0)$ is increasing for $k \geq k_0$? We show that the answer in “no”.

We assume that

$$\Delta_k(\mu_0) \leq \Delta_{k+1}(\mu_0) \quad \text{for} \quad k \geq k_0.$$ 

This is equivalent to, say

$$-\mu_0 \leq \frac{\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k}}{p_{k+1} - p_k} = \sigma_k. \quad (3.1)$$

Using the asymptotic formula

$$\log \binom{k^2}{k} = \left( k - \frac{1}{2} \right) \log k + k - \frac{1}{2} (1 + \log(2\pi)) + O(1/k),$$

see [4], we get

$$\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k} = \log k + 2 + O(1/k). \quad (3.2)$$

Applying (3.2) and the limit relations

$$\lim_{k \to \infty} \frac{\log p_k}{\log k} = 1 \quad \text{and} \quad \lim_{k \to \infty} \inf \frac{\log p_k}{p_{k+1} - p_k} = 0,$$
see [8], [10], yields

\[ \liminf_{k \to \infty} \sigma_k = \liminf_{k \to \infty} \left( \frac{\log p_k}{p_{k+1} - p_k} \right) = \liminf_{k \to \infty} \left( \frac{\log k + 2 + O(1/k)}{\log p_k} \right) = 0. \]

From (3.1) and (3.3) we conclude that \( \mu_0 \geq 0 \).

Acknowledgements. The numerical calculations were carried out by Professor H. J. J. Te Riele (Centrum Wiskunde & Informatica, Amsterdam) by using the computer program GP/PAR I. We are very grateful for his support.

REFERENCES


7. G. Robin: Estimation de la fonction de Tchebychev \( \theta \) sur le \( k \)-ième nombre premier et grandes valeurs de la fonction \( \omega(n) \), nombre de diviseurs premiers de \( n \). Acta Arith., 42 (1983), 367–389.


Morsbacher Str. 10, 51545 Waldbröl, Germany
E-mail: H.Alzer@gmx.de

Babes-Bolyai University, Department of Mathematics, Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania
E-mail: jsandor@math.ubbcluj.ro