ASYMPTOTIC EXPANSION FOR THE SUM OF INVERSES OF ARITHMETICAL FUNCTIONS INVOLVING ITERATED LOGARITHMS

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A generalized formula is obtained for the sum of inverses of the prime counting function for a large class of arithmetical functions related to the iterated logarithms.

1. INTRODUCTION AND MAIN RESULT

Let \( \pi(x) \) be the number of primes not exceeding \( x \). In 2000, using the asymptotic formula

\[
\pi(x) = \frac{x}{\log(x)} \left( \sum_{k=0}^{m-1} \frac{k!}{\log^k(x)} + O\left( \frac{1}{\log^m(x)} \right) \right),
\]

L. Panaitopol [4] obtained

\[
\frac{1}{\pi(x)} = \frac{1}{x} \left( \log(x) - 1 \frac{k_1}{\log(x)} - \cdots - \frac{k_m}{\log^m(x)} + O\left( \frac{1}{\log^{m+1}(x)} \right) \right),
\]

where \( m \geq 1 \) and \( \{k_j\}_j \) is the sequence of integers given by the recurrence relation

\[ k_n + 1!k_{n-1} + 2!k_{n-2} + \cdots + (n-1)!k_1 = n \cdot n! \,.
\]
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Two years later, A. Ivic [3] proved that

$$\sum_{2 \leq n \leq x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2(x) - \log(x) - \log \log(x) + C$$

$$+ \frac{k_2}{\log(x)} + \cdots + \frac{k_m}{(m-1) \log^{m-1}(x)} + O\left(\frac{1}{\log^m(x)}\right),$$

where $C$ is an absolute constant not depending on $m$.

In 2009, the first author and F. Bencherif [1] derived an asymptotic formula for the sum of reciprocals of a large class of arithmetic functions having the following expansion

$$f(n) = \frac{n}{\log(n)} \left( a_0 + \frac{a_1}{\log(n)} + \cdots + \frac{a_{m-1}}{\log^{m-1}(n)} + O\left(\frac{1}{\log^m(n)}\right) \right),$$

with $a_0 \neq 0$, they obtained

$$\sum_{2 \leq n \leq x} \frac{1}{f(n)} = \frac{b_0}{2} \log^2(x) + b_1 \log(x) + b_2 \log \log(x) + C_0$$

$$- \frac{b_3}{\log(x)} - \cdots - \frac{b_{m+1}}{(m-1) \log^{m-1}(x)} + O\left(\frac{1}{\log^m(x)}\right),$$

where $\sum_{2 \leq n \leq x} \frac{1}{f(n)}$ is a sum restricted to integers $n$ for which $f(n) \neq 0$ and $b_j = A_j(a_0, a_1, \ldots, a_j)$ for $0 \leq j \leq m + 1$, with

$$A_0(t_0) = \frac{1}{t_0}, \quad A_1(t_0, t_1) = -\frac{t_1}{t_0^2},$$

$$A_n(t_0, t_1, \ldots, t_n) = \frac{(-1)^n}{t_0^{n+1}},$$

where

$$A_n(t_0, t_1, \ldots, t_n) = \frac{(-1)^n}{t_0^{n+1}}.$$

More recently, the authors in [2] studied the arithmetical function $nK(n)$, where

$$K(x) := \max \{ k \in \mathbb{N} / p_1 p_2 \cdots p_k \leq x \},$$

and $p_k$ is the $k^{th}$ prime number. Using the asymptotic expansion

$$K(x) = \frac{\log(x)}{\log \log(x)} \left( \sum_{j=0}^{m} \frac{j!}{[\log \log(x)]^j} + O\left(\frac{1}{[\log \log(x)]^{m+1}}\right) \right),$$
they get a similar result to the one in A. Ivč [3], with three levels of logarithmic iterations \( x, \log x, \log \log x, \)

\[
\sum_{2 \leq n \leq x} \frac{1}{nK(n)} = \frac{1}{2} \log^2 \log(x) - \log \log(x) - \log \log \log(x) + C_1
\]

\[+ \frac{k_2}{\log \log(x)} + \cdots + \frac{k_m}{(m-1) \log^{m-1} \log(x)} + O \left( \frac{1}{\log^m \log(x)} \right), \]

where \( C_1 \) is an absolute constant not depending on \( m \).

Let \( s \geq 0 \) be an integer. We define the function

\[
L_s(x) := \prod_{i=0}^{s} \log_i(x), \quad \text{with} \quad \log_i(x) = \underbrace{\log \cdots \log}_{i \text{ times}}(x) \text{ and } \log_0(x) = x.
\]

For \( s = 2 \), \( L_2(x) = x \log(x) \log \log(x) \).

Let \( f_s \) be an arithmetical function admitting, for all \( m \geq 1 \), the following asymptotic formula

\[
f_s(n) = \frac{L_s(n)}{\log_{s+1}(n)} \left\{ \sum_{i=0}^{m-1} \frac{a_i}{\log_{s+1}^i(n)} + O \left( \frac{1}{\log_{s+1}^m(n)} \right) \right\}, \quad a_0 \neq 0.
\]

For \( s = 0 \) and \( a_i = i! \), we obtain (1), which corresponds to \( \pi(n) \). For \( s = 1 \) with \( a_i = i! \), we find (2), which corresponds to \( nK(n) \).

Considering the above background, here is our main result:

**Theorem 1.** For all integers \( m \geq 1 \) and \( s \geq 0 \), we have

\[
\sum_{n \leq x} \frac{1}{f_s(n)} = \frac{\delta_0}{2} \log_{s+1}^2(x) + \delta_1 \log_{s+1}(x) + \delta_2 \log_{s+2}(x) + C_s
\]

\[- \frac{\delta_3}{\log_{s+1}(x)} - \cdots - \frac{\delta_{m+1}}{(m-1) \log_{s+1}^{m-1}(x)} + O \left( \frac{1}{\log_{s+1}^m(x)} \right),
\]

where \( \sum_{n \leq x} \frac{1}{f_s(n)} \) is a sum restricted to integers \( e(s) < n \leq x \) for which \( f_s(n) \neq 0 \), \( C_s \) is an absolute constant not depending on \( m \), \( \{\delta_i\} \) is the sequence given by the recurrence relation

\[a_0 \delta_n + a_1 \delta_{n-1} + \cdots + a_{m} \delta_0 = 0, \quad a_0 \delta_0 = 1,
\]

and \( e(s) := \exp \exp \cdots \exp (0) \).

For \( a_i = i! \) and \( s = 0 \) and \( s = 1 \), respectively we find the results of A. Ivč [3] and H. Belbachir and D. Berkane [2].
2. LEMMAS AND PROOF OF THE MAIN RESULT

Let \( \{ \delta_i \} \) be the sequence of real numbers defined by expanding the following expression of the rational function \( \Delta \), for \( y > 0 \) we consider

\[
\Delta(y) := \left( \sum_{i=0}^{m} \frac{a_i}{y^{i+1}} \right) \left( \sum_{i=0}^{m+1} \frac{\delta_i}{y^{i-1}} \right), \quad m \geq 1,
\]

such that \( a_0 \delta_0 = 1 \), and terms with \( \frac{1}{y^i}, 1 \leq i \leq m \) vanish.

Then, when \( y \to \infty \), we obtain

\[
(4) \quad \Delta = 1 + O\left( \frac{1}{y^{m+1}} \right).
\]

**Lemma 1.** The coefficient \( \delta_n, n \geq 1 \), is given by the relation

\[
\delta_n = \frac{1}{a_{n+1}^{m+1}} \begin{vmatrix}
0 & a_1 & \ldots & a_{n-1} & a_n \\
0 & a_0 & \ldots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & a_0 & a_1 \\
1 & 0 & \ldots & 0 & a_0
\end{vmatrix}
\]

**Proof.** From the definition of \( \Delta(y) \), we notice that the vector \( \delta = (\delta_0, ..., \delta_n) \), is the unique solution to the following Cramer’s system

\[
\begin{align*}
a_0 \delta_n + a_1 \delta_{n-1} + \cdots + a_n \delta_0 &= 0 \\
a_0 \delta_{n-1} + \cdots + a_{n-1} \delta_0 &= 0 \\
\vdots \quad & \quad \vdots \\
a_0 \delta_1 + a_1 \delta_0 &= 0 \\
a_0 \delta_0 &= 1.
\end{align*}
\]

**Lemma 2.** For \( n \) sufficiently large, we have

\[
f_s(n) = \frac{L_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \varepsilon(n)},
\]

where \( \lim_{n \to \infty} \varepsilon(n) = 0 \).

**Proof.** From (3), we have

\[
(5) \quad f_s(n) = L_s(n) \left( \sum_{j=0}^{m} \frac{a_j}{\log_{j+1}^s(n)} \right) + O\left( \frac{L_s(n)}{\log_{s+1}^m(n)} \right),
\]

and from (4) it follows

\[
(6) \quad \sum_{j=0}^{m} \frac{a_j}{y^{j+1}} = \frac{1 + O\left( \frac{1}{y^{m+1}} \right)}{\delta_0 y + \sum_{i=1}^{m+1} \frac{\delta_i}{y^{i-1}}} = \frac{1}{\delta_0 y + \sum_{i=1}^{m+1} \frac{\delta_i}{y^{i-1}}} + O\left( \frac{1}{y^{m+2}} \right).
\]
The substitution of $y = \log_{s+1}(n)$ in (6) and in relation (5) gives

$$f_s(n) = \frac{L_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}}{\log^m_{s+1}(n)}} + O\left(\frac{L_s(n)}{\log^m_{s+1}(n)}\right).$$

Thus we can write

$$f_s(n) = \frac{L_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \varepsilon(n)},$$

with $\varepsilon(n) = O\left(\frac{1}{\log_{s+1}(n)}\right)$ from which it follows that $\lim_{n \to \infty} \varepsilon(n) = 0$.

The case $s = 0$ and $a_i = i!$, gives the approximation given by L. Panaitopol [4],

$$\pi(n) = \frac{n}{\log(n) - 1 - \varepsilon(n)}.$$  \(\square\)

**Proof of the main result.** Simplifying formula (7), we can write for all $m \geq 1$,

$$f_s(n) = \frac{L_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}(1 + \varepsilon_m(n))}{\log^m_{s+1}(n)}},$$

with

$$\varepsilon_m(n) \ll_m \frac{1}{\log_{s+1}(n)}.$$

Then, for all $m \geq 1$ and all $n > \epsilon(s)$, we obtain

$$\frac{1}{f_s(n)} = \frac{1}{L_s(n)} \left(\frac{\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)}}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}(1 + \varepsilon_m(n))}{\log^m_{s+1}(n)}\right),$$

and by summation, we obtain

$$\sum_{n \leq x} \frac{1}{f_s(n)} = A_1 + A_2 + A_3 + \sum_{r=2}^{m} B_r + \sum_{\epsilon(s) < n \leq x} \frac{\delta_{m+1}\varepsilon_m(n)}{L_s(n) \log^m_{s+1}(n)},$$

with

$$A_1 = \sum_{\epsilon(s) < n \leq x} \frac{\delta_0 \log_{s+1}(n)}{L_s(n)}, \quad A_2 = \sum_{\epsilon(s) < n \leq x} \frac{\delta_1}{L_s(n)},$$

$$A_3 = \sum_{\epsilon(s) < n \leq x} \frac{\delta_2}{L_s(n) \log_{s+1}(n)}, \quad B_r = \sum_{\epsilon(s) < n \leq x} \frac{\delta_{r+1}}{L_s(n) \log^r_{s+1}(n)}, \quad 2 \leq r \leq m.$$
Let us evaluate these sums. First we can notice that the functions involved in the previous sums are all positive and decreasing for a given constant $\omega \geq e(s)$.

Let’s compose for $A_1$,

$$\sum_{|\omega|<n\leq x} \frac{\log_{s+1}(n)}{E_s(n)} = \int_{|\omega|}^{x} \frac{\log_{s+1}(t)}{E_s(t)} \, dt + O\left(\frac{\log_{s+1}(t)}{E_s(t)}\right).$$

Thus there is a constant $\alpha_1$ which includes the sum $\sum_{n=2}^{\lfloor x \rfloor} \frac{\log_{s+1}(n)}{E_s(n)}$ such that

$$A_1 = \frac{\delta_0}{2} \log_{s+1}^2(x) + \alpha_1 + O\left(\frac{\log_{s+1}(x)}{E_s(x)}\right).$$

Using similar argument, we also obtain

$$A_2 = \delta_1 \log_{s+1}(x) + \alpha_2 + O\left(\frac{1}{E_s(x)}\right),$$

$$A_3 = \delta_2 \log_{s+2}(x) + \alpha_3 + O\left(\frac{1}{E_s(x) \log_{s+1}(x)}\right),$$

$$B_r = \frac{-\delta_{r+1}}{(r-1) \log_{s+1}^{r-1}(x)} + \beta_r + O\left(\frac{1}{E_s(x) \log_{s+1}^r(x)}\right).$$

As $\epsilon_m(n)$ is bounded and the series

$$\sum_{n>\epsilon(s)} \frac{1}{E_s(n) \log_{s+1}^m(n)},$$

is convergent for all $m \geq 2$ (Bertrand’s series), with the sum noted $S_m$, we deduce that

$$\sum_{\epsilon(s)<n\leq x} \frac{\delta_{m+1} \epsilon_m(n)}{E_s(n) \log_{s+1}^m(n)} = S_m + O\left(\frac{1}{\log_{s+1}^m(x)}\right).$$

Putting together the above expression in (8) we infer that

$$\sum_{n\leq x} \frac{1}{f_s(n)} = \frac{\delta_0}{2} \log_{s+1}^2(x) + \delta_1 \log_{s+1}(x) + \delta_2 \log_{s+2}(x) + \alpha_1 + \alpha_2 + \alpha_3 + \sum_{r=2}^{m} \beta_r + S_m$$

$$- \frac{\delta_3}{\log_{s+1}(x)} - \ldots - \frac{\delta_{m+1}}{(m-1) \log_{s+1}^{m-1}(x)} + O\left(\frac{1}{\log_{s+1}^m(x)}\right).$$

Setting $C_s = \alpha_1 + \alpha_2 + \alpha_3 + \sum_{r=2}^{m} \beta_r + S_m$ we find the formula mentioned in the main Theorem. This constant is independent of the value of $m$ because the difference between two developments of $\sum_{n\leq x} \frac{1}{f_s(n)}$ is a quantity which is absorbed by the roundness when $x \to +\infty$. \qed
Acknowledgments. The authors wish to warmly thank their referee for valuable advice and comments which helped to improve the quality of this paper.

REFERENCES