RELAXATIONS OF HALL’S CONDITION: OPTIMAL BATCH CODES WITH MULTIPLE QUERIES

Csilla Bujtás, Zsolt Tuza

Combinatorial batch codes model the storage of a database on a given number of servers such that any $k$ or fewer items can be retrieved by reading at most $t$ items from each server. A combinatorial batch code with parameters $n, k, m, t$ can be represented by a system $\mathcal{F}$ of $n$ (not necessarily distinct) sets over an $m$-element underlying set $X$, such that for any $k$ or fewer members of $\mathcal{F}$ there exists a system of representatives in which each element of $X$ occurs with multiplicity at most $t$. The main purpose is to determine the minimum $N(n, k, m, t)$ of total data storage $\sum_{F \in \mathcal{F}} |F|$ over all combinatorial batch codes $\mathcal{F}$ with given parameters.

Previous papers concentrated on the case $t = 1$. Here we obtain the first nontrivial results on combinatorial batch codes with $t > 1$. We determine $N(n, k, m, t)$ for all cases with $k \leq 3t$, and also for all cases where $n \geq t\left(\left\lceil \frac{m}{k/t} \right\rceil - 2\right)$. Our results can be considered equivalently as minimum total size $\sum_{F \in \mathcal{F}} |F|$ over all set systems $\mathcal{F}$ of given order $m$ and size $n$, which satisfy a relaxed version of Hall’s Condition; that is, $|\bigcup F'| \geq |F'|/t$ holds for every subsystem $F' \subseteq \mathcal{F}$ of size at most $k$.

1. INTRODUCTION

Combinatorial batch codes and dual systems. Batch codes were introduced by Ishai, Kushilevitz, Ostrovsky and Sahai [10]. They represent the distributed storage of an $n$-element database on a set of $m$ servers when any $k$ or fewer data items can be recovered by submitting a limited number $t$ of queries to each server. This model can be used for amortizing the computational cost in

2010 Mathematics Subject Classification. 05D05, 05C65, 68R05.

Keywords and Phrases. Combinatorial batch code, dual system, Hall-type condition, system of representatives.
private information retrieval. Combinatorial batch code, studied in detail first by Paterson, Stinson and Wei [13], is the version of a batch code in which each server stores a subset of the database and decoding simply means reading items from servers. The latter model admits a purely combinatorial definition as a set system satisfying a requirement on systems of representatives. Therefore, it is in close connection with Hall-type conditions.

A set system $\mathcal{F}$ over an underlying set $X$ is the collection of some nonempty subsets of $X$. Objects $x \in X$ are called elements whilst objects $F \in \mathcal{F}$ are referred to as members. Moreover, the order and the size of a system $\mathcal{F}$ are the number $|X|$ of elements and the number $|\mathcal{F}|$ of members, respectively. The total size of a system $\mathcal{F}$ is defined as $\sum_{F \in \mathcal{F}} |F|$. Throughout this paper, ‘set system’ is meant as a ‘multisystem’; that is, repetitions are allowed, distinct members of the system may correspond to the same subset of the underlying set.

A combinatorial batch code with parameters $n, k, m, t$ can be represented with its ‘dual’ set system (shortly, CBC$(n, k, m, t)$-system) $\mathcal{F}$, where the $m$ elements of the underlying set correspond to the $m$ servers and the members of $\mathcal{F}$ correspond to the $n$ items of data. A member $F_i \in \mathcal{F}$ then means the set of servers where the $i$th data item is stored. Hence, the total amount of data collectively stored by the $m$ servers—which is the object of minimization—equals the total size of system $\mathcal{F}$.

**Definition 1.** For positive integers $k$ and $t$, a set system $\mathcal{F}$ is a CBC$(k, t)$-system if, for every subsystem $\mathcal{F}' = \{F_1, \ldots, F_\ell\} \subseteq \mathcal{F}$ of size $1 \leq \ell \leq k$, there exist elements $x_1, \ldots, x_\ell$ such that $x_i \in F_i$ holds for every $1 \leq i \leq \ell$ and each element of $X$ has multiplicity at most $t$ in $\{x_1, \ldots, x_\ell\}$. A set system $\mathcal{F}$ over the underlying set $X$ is called a CBC$(n, k, m, t)$-system if $|\mathcal{F}| = n$, $|X| = m$, and $\mathcal{F}$ is a CBC$(k, t)$-system. Moreover, $N(n, k, m, t) := \min_{\mathcal{F}} \sum_{F \in \mathcal{F}} |F|$ denotes the minimum total size of a system taken over all CBC$(n, k, m, t)$-systems $\mathcal{F}$, subject to that there exists at least one such system.

Note that if both $mt < k$ and $mt < n$ hold, no CBC$(n, k, m, t)$-system exists. Otherwise, the system containing the underlying set $X$ as member with multiplicity $n$ is a CBC$(n, k, m, t)$ and hence $N(n, k, m, t)$ is well-defined. We will assume throughout that $n, k, m$ and $t$ denote positive integers such that $mt \geq \min\{n, k\}$. Systems which are CBC$(n, k, m, t)$ and have minimum total size $N(n, k, m, t)$ will be called optimal.

**Hall-type conditions.** Hall’s Theorem [9] and related results on algorithms serve as basic tools in several branches of combinatorics and discrete optimization. Also, nonstandard Hall-type conditions and their consequences were intensively studied (see, e.g., [6, 7, 8, 11, 12]). Each earlier paper on combinatorial batch codes with $t = 1$ applied Hall’s Condition. Here we use a relaxed version whose origin goes back to the works [7, 8, 12].
Definition 2. We say that a set system $F$ satisfies the $(k, t)$-Hall Condition (shortly, $(k, t)$-HC) if $|\bigcup F'| \geq |F'|/t$ holds for every subsystem $F' \subseteq F$ which contains at most $k$ members.

Results. In [1, 2, 3, 4, 10, 13] several results on combinatorial batch codes were obtained, moreover their connections with transversal matroids [2], unbalanced expander graphs [10] and binary constant-weight codes [1] were also pointed out. These papers considered—nearly exclusively—the case of $t = 1$, although some simple relations between combinatorial batch codes with $t > 1$ and those with $t = 1$ were established in [10].

In this paper we obtain the first nontrivial results for the case of general $t$. In Section 2 we prove the Equivalence Theorem, which is a three-sided characterization: beside the equivalence of the $(k, t)$-Hall Condition and the property of being a CBC$^*$$(k, t)$-system, the requirement can also be expressed in a form which implies that if $[k/t] = [k'/t]$ then a CBC$^*$$(k, t)$-system is a CBC$^*$$(k', t)$-system and vice versa. Some further basic properties and a cardinality-balancing transformation will be presented, too. In Section 3 and Section 4 we determine the minimum total size $N(n, k, m, t)$ for all parameters satisfying $n \geq t\left(\frac{m}{[k/t]} - 2\right)$ and for all cases where $k \leq 3t$, respectively. By the Equivalence Theorem, several methods developed originally for the case $t = 1$ can be applied for the general setting $t \geq 1$. Our proof techniques used here are similar to those in [3] and occasionally to those in [1] and [13], too. Some results proved here have been announced without proofs in [5].

2. SOME BASIC PROPERTIES

In this section we deal with three types of properties. First, we give three equivalent conditions for a system to be a CBC$^*$$(k, t)$. Then, we present some basic inequalities about the size distributions of members in a CBC$^*$$(n, k, m, t)$, and finally we show that for every four-tuple of parameters there exists an optimal CBC$^*$$(n, k, m, t)$ which either does not contain members larger than $[k/t] - 1$ or does not contain members smaller than $[k/t] - 1$.

In the following theorem, the equivalence of (i) and (ii) is a consequence of more general results on systems of representatives [8, 12, 7], hence we prove only the equivalence of (ii) and (iii).

Theorem 3. (Equivalence Theorem) For all positive integers $k$ and $t$, and for every set system $F$, the following statements are equivalent:

(i) $F$ is a CBC$^*$$(k, t)$-system.

(ii) $F$ satisfies the $(k, t)$-Hall Condition.

(iii) For every $\ell < [k/t]$ and for every $\ell$-element subset $X'$ of the underlying set, at most $\ell t$ members of $F$ are subsets of $X'$.
Proof. (ii) $\Leftrightarrow$ (iii) We prove the equivalence of the negations of (ii) and (iii). If (ii) does not hold, there exists a subsystem $\mathcal{F}' \subseteq \mathcal{F}$ of size $i \leq k$, for which the union $X' = \bigcup \mathcal{F}'$ has at most $\lceil i/t \rceil - 1$ elements. That is, $X'$ contains at least $i > t \left( \lfloor i/t \rfloor - 1 \right) \geq t \cdot X'$ members of $\mathcal{F}$, and also $|X'| \leq \lceil k/t \rceil - 1$ is valid. This means that (iii) does not hold either. From the other direction, if a subset $X' \subseteq X$ of cardinality $\ell \leq \lceil k/t \rceil - 1$ contains more than $\ell t$ members from $\mathcal{F}$, then the union of any $\ell t + 1 \leq k$ of these members can contain at most $|X'| = \ell < \ell t + \left( \ell t + 1/t \right)$ elements, which contradicts (ii).

Part (iii) of Theorem 3 expresses the $(k, t)$-Hall Condition referring only to $[k/t]$ and $t$ as parameters. Hence, if an integer $t > 1$ is fixed, not the exact value of $k$ but only $[k/t]$ is that really matters the meaning of $(k, t)$-HC. Particularly, it would suffice to determine the optimal total size $N(n, k, m, t)$ only for cases where $k$ is divisible by $t$.

Corollary 4. Assume that $[k/t] = [k'/t]$. Then, a system $\mathcal{F}$ is a $CBC^*(k, t)$-system if and only if it is a $CBC^*(k', t)$-system; moreover, $\mathcal{F}$ satisfies the $(k, t)$-Hall Condition if and only if it satisfies the $(k', t)$-Hall Condition. Particularly, if $[k/t] = [k'/t]$ then $N(n, k, m, t) = N(n, k', m, t)$ is valid for all $n$ and $m$.

From now on, also requirement (iii) from the Equivalence Theorem will be referred to as $(k, t)$-HC. Applying Theorem 3, the next necessary condition for systems satisfying $(k, t)$-HC is easy to verify. The analogous result for the special case of $t = 1$ first appeared in a proof of [13], and later it was stated in [1] and [3] as well.

Theorem 5. Let $\mathcal{F}$ be a $CBC^*(n, k, m, t)$ and let $\ell_i$ denote the number of $i$-element members of $\mathcal{F}$, for every $1 \leq i \leq \lceil k/t \rceil$. Then,

$$\sum_{i=1}^{\lfloor k/t \rfloor - 1} \ell_i \left( \binom{m-i}{\lceil k/t \rceil - 1 - i} \right) \leq t \left( \binom{k}{t} - 1 \right) \binom{m}{\lceil k/t \rceil - 1}.$$

Proof. We are going to estimate the number $z$ of pairs $(F, A)$ with $F \in \mathcal{F}$, $F \subseteq A \subseteq X$ and $|A| = \lceil k/t \rceil - 1$. Every $i$-element member $F$ from $\mathcal{F}$ is contained in exactly $\binom{m-i}{\lceil k/t \rceil - 1 - i}$ such subsets $A$. Consequently, $z = \sum_{i=1}^{\lfloor k/t \rfloor - 1} \ell_i \binom{m-i}{\lceil k/t \rceil - 1 - i}$. On the other hand, since $\mathcal{F}$ satisfies $(k, t)$-HC, every $((\lceil k/t \rceil - 1)$-element $A \subseteq X$ contains at most $t((\lceil k/t \rceil - 1)$ members from $\mathcal{F}$. Therefore, $z \leq t((\lceil k/t \rceil - 1) \binom{m}{\lceil k/t \rceil - 1}$ and the inequality stated in the theorem follows.

Corollary 6. Every $CBC^*(n, k, m, t)$ contains at most $t((\lceil k/t \rceil - 1) \binom{m}{\lceil k/t \rceil - 1}$ members of size not exceeding $\lceil k/t \rceil - 1$.

Due to the Equivalence Theorem, we can take some observations on extensions of a $CBC^*(k, t)$-system $\mathcal{F}$ with a new member $F \subseteq X$. First, since the fulfi-
ment of \((k, t)\)-HC depends only on members of size at most \([k/t]−1\), the following statement clearly holds.

**Observation 7.** If \(\mathcal{F}\) is a CBC\((k, t)\)-system and \(|\mathcal{F}| \geq [k/t]\), then \(\mathcal{F} \cup \{F\}\) is a CBC\((k, t)\)-system, as well. Therefore, an optimal CBC\((n, k, m, t)\)-system does not contain members of size greater than \([k/t]\).

Second, since a member \(F\) of size \([k/t]−1\) is not contained in a \(([k/t]−1)\)-element subset of \(X\) other than itself, the following statement is valid.

**Proposition 8.** Let \(\mathcal{F}\) be a CBC\((k, t)\)-system and \(|\mathcal{F}| = [k/t]−1\). Then, \(\mathcal{F} \cup \{F\}\) is a CBC\((k, t)\)-system if and only if \(F\) contains fewer than \(t([k/t]−1)\) members from \(\mathcal{F}\). Moreover, if \(\ell_i\) denotes the number of members of size \(i\) in \(\mathcal{F}\) (for each \(1 \leq i \leq [k/t]−1\)), then \(\mathcal{F}\) can be extended with \(L\) appropriately chosen new members each of cardinality \([k/t]−1\), such that the system remains a CBC\((k, t)\), if and only if

\[
L \leq t\left(\left\lceil \frac{k}{t} \right\rceil−1\right)\left(\frac{m}{[k/t]−1}\right)−\sum_{i=1}^{[k/t]−1} \ell_i\left(\frac{m−i}{[k/t]−1−i}\right).
\]

Next, we present a transformation which is applicable for two members of a CBC\((n, k, m, t)\) if one of them contains the other. Then, some (any) elements from the larger member can be transferred to the smaller one and the system remains a CBC\((n, k, m, t)\) with the same total size. This transformation was introduced in [3] (Proposition 1) for the case \(t = 1\). In fact the proof remains the same for the general case \(t \geq 1\), hence it is omitted here.

**Proposition 9.** [3] Let \(\mathcal{F}\) be a CBC\((n, k, m, t)\) with two members \(F_1 \subset F_2\) for which \(|F_1| + 2 \leq |F_2|\) and let \(A\) be a nonempty set such that \(A \subset F_2 \setminus F_1\). Then, replacing \(F_1\) and \(F_2\) with \(F'_1 = F_1 \cup A\) and \(F'_2 = F_2 \setminus A\), the obtained system \(\mathcal{F}'\) is a CBC\((n, k, m, t)\) as well, and the two systems \(\mathcal{F}\) and \(\mathcal{F}'\) have the same total size.

We say that a CBC\(\ast\) is of type \([a, b]\) if the size of each \(F \in \mathcal{F}\) satisfies \(a \leq |F| \leq b\). Due to Observation 7, every optimal CBC\((n, k, m, t)\)-system is of type \([1, [k/t]]\). By Proposition 9 we can prove a stronger result for \([k/t] \geq 3\).

**Proposition 10.** If \([k/t] \geq 3\), then for every optimal CBC\((n, k, m, t)\)-system \(\mathcal{F}\), there exists an \(\mathcal{F}'\) which is an optimal CBC\((n, k, m, t)\) as well, and has type either \([1, [k/t]−1]\) or \([[[k/t]−1, [k/t]]\).

**Proof.** Suppose that an optimal CBC\((n, k, m, t)\)-system \(\mathcal{F}\) contains a member \(F_1\) of size \([k/t]−2\) and also a member \(F_2\) of size \([k/t]\). Observation 7 implies that \(F_2\) can be replaced with any \([k/t]\)-element subset \(F'_2\) of the underlying set. Let us choose this new member such that \(F'_2 \supset F_1\). Now, applying the transformation described in Proposition 9, an optimal CBC\((n, k, m, t)\)-system \(\mathcal{F}'\) is obtained which contains fewer members of size \([k/t]\) than \(\mathcal{F}\) did. Repeated application of this procedure yields an optimal CBC\((n, k, m, t)\) of type either \([1, [k/t]−1]\) or \([[[k/t]−1, [k/t]]\).

\(\square\)
In the simple cases listed in the following observation it is enough to take \( n \) singletons to obtain a CBC\(^*\)(\( n, k, m, t \)).

**Observation 11.** If at least one of \( n \leq tm \) and \( k \leq t \) is valid, then \( N(n, k, m, t) = n \).

The next proposition is the generalization of Theorem 4 of [13].

**Proposition 12.** For every four positive integers \( n, k, m \) and \( t \), if \( m = \lceil k/t \rceil \) and \( n \geq tm \), then \( N(n, k, m, t) = mn - tm(m - 1) \).

**Proof.** Under the given conditions consider a CBC\(^*\)(\( n, k, m, t \))-system \( F \). By \((k, t)\)-HC, for every element \( x \) of the underlying set \( X \), the \((m - 1)\)-element set \( X \setminus \{x\} \) covers entirely at most \( t(m - 1) \) members of \( F \). Thus, \( x \) has to be involved in at least \( n - t(m - 1) \) members of \( F \). Therefore, counting the total size of the system by summing up the degrees of elements, \( N(n, k, m, t) \geq m(n - t(m - 1)) \) must hold.

On the other hand, let \( F^* \) be the system over the underlying set \( X = \{x_1, \ldots, x_m\} \), in which \( X \) is a member with multiplicity \( n - tm \) and each singleton \( \{x_i\} \) occurs with multiplicity \( t \). Clearly, \( F^* \) is a CBC\(^*\)(\( n, k, m, t \))-system and its total size is exactly \( tm + (n - tm)m = mn - tm(m - 1) \). This verifies the statement.

### 3. OPTIMUM VALUES FOR \( n \geq t \left( \left\lceil \frac{k}{t} \right\rceil - 2 \right) \)

**Theorem 13.** If \( m \geq \left\lceil \frac{k}{t} \right\rceil \) and \( n > t \left( \left\lceil \frac{k}{t} \right\rceil - 1 \right) \left( \left\lfloor \frac{m}{\left\lceil k/t \right\rceil - 1} \right\rfloor \right) \), then

\[
N(n, k, m, t) = n \left\lceil \frac{k}{t} \right\rceil - t \left( \left\lceil \frac{k}{t} \right\rceil - 1 \right) \left( \left\lfloor \frac{m}{\left\lceil k/t \right\rceil - 1} \right\rfloor \right).
\]

**Proof.** Consider parameters \( n, k, m \) and \( t \) satisfying the conditions given in the theorem. Due to Corollary 6, the number of members of \( F \) which are of size smaller than \( \left\lceil k/t \right\rceil \) is at most \( t \left( \left\lceil k/t \right\rceil - 1 \right) \left( \left\lfloor m/\left\lceil k/t \right\rceil - 1 \right\rfloor \right) \). Thus, under the present conditions, system \( F \) cannot be of type \([1, \left\lceil k/t \right\rceil - 1] \). Then, Proposition 10 implies that there exists an optimal CBC\(^*\)(\( n, k, m, t \))-system \( F \) of type \([\left\lceil k/t \right\rceil - 1, \left\lceil k/t \right\rceil] \). The total size of \( F \) is precisely \( n \left\lceil k/t \right\rceil - n' \) where \( n' \) denotes the number of \( \left( \left\lceil k/t \right\rceil - 1 \right) \)-element members. Applying Corollary 6 again, we obtain

\[
N(n, k, m, t) = n \left\lceil \frac{k}{t} \right\rceil - n' \geq n \left( \left\lceil \frac{k}{t} \right\rceil - t \left( \left\lceil \frac{k}{t} \right\rceil - 1 \right) \left( \left\lfloor \frac{m}{\left\lceil k/t \right\rceil - 1} \right\rfloor \right) .
\]

On the other hand, take each \( \left( \left\lceil k/t \right\rceil - 1 \right) \)-element subset of an \( m \)-element underlying set \( X \) with multiplicity \( t \left( \left\lceil k/t \right\rceil - 1 \right) \) and further \( n - t \left( \left\lceil k/t \right\rceil - 1 \right) \left( \left\lfloor m/\left\lceil k/t \right\rceil - 1 \right\rfloor \right) \) subsets
of $X$, each of cardinality $\lceil k/t \rceil$. This construction is clearly a CBC$^\ast(n, k, m, t)$-
-system and proves that $N(n, k, m, t) \leq n\lceil k/t \rceil - t(\lceil k/t \rceil - 1)\left(\frac{m}{\lceil k/t \rceil - 1}\right)$. This verifies the theorem.

To obtain a formula for the second highest range of $n$, we will apply the following technical lemma proved in [3].

**Lemma 14.** [3] For any three integers $i, p, m$, if $1 \leq i \leq p \leq m - 1$, then

$$\left\lfloor \frac{(m - i)}{(p - i)} \right\rfloor \geq p - i.$$ 

**Theorem 15.** If $m \geq \lceil k/t \rceil \geq 3$ and $t\left(\frac{m}{\lceil k/t \rceil - 2}\right) \leq n \leq t\left(\left\lfloor \frac{k}{t} \right\rfloor - 1\right)\left(\frac{m}{\lceil k/t \rceil - 1}\right)$, then

$$N(n, k, m, t) = n\left(\left\lfloor \frac{k}{t} \right\rfloor - 1\right) - \left\lfloor \frac{t\left(\left\lfloor \frac{k}{t} \right\rfloor - 1\right)\left(\frac{m}{\lceil k/t \rceil - 1}\right) - n}{m - \frac{k}{t} + 1}\right\rfloor.$$ 

**Proof.** If $m = \lceil k/t \rceil$, the statement yields $N(n, k, m, t) = mn - tm(m - 1)$ which corresponds to Proposition 12. Hence, we assume that $m > \lceil k/t \rceil$. Let us introduce the notation

$$K := \left\lfloor \frac{k}{t} \right\rfloor, \quad y := \left\lfloor \frac{t(K - 1)\left(\frac{m}{K - 1}\right) - n}{m - K + 1}\right\rfloor.$$ 

We construct a CBC$^\ast(n, k, m, t)$-system $F^\ast$ on an $m$-element underlying set $X$ as follows. First, choose $y$ sets, each of cardinality $K - 2$, such that every $(K - 2)$-element subset of $X$ has multiplicity at most $t$. This can be done, since by the given condition, $t\left(\frac{m}{K - 2}\right) \leq n$ holds and hence,

$$y \leq \frac{t(K - 1)\left(\frac{m}{K - 1}\right) - n}{m - K + 1} \leq \frac{t(m - K + 2)\left(\frac{m}{K - 2}\right) - t\left(\frac{m}{K - 2}\right)}{m - K + 1} = t\left(\frac{m}{K - 2}\right).$$ 

Since every $(K - 2)$-element subset of $X$ contains at most $t$ members, and every $(K - 1)$-element subset contains at most $t(K - 1)$ members, the obtained system is a CBC$^\ast(k, t)$. Moreover, in view of Proposition 8, the following inequality proves that the system can be extended with $n - y$ members, each of cardinality $K - 1,$
such that a CBC\(^*\)(n, k, m, t)-system \(F^*\) is obtained.

\[
t (K - 1) \left(\frac{m}{K - 1}\right) - \left\lfloor \frac{t(K - 1) \left(\frac{m}{K - 1}\right) - n}{m - K + 1} \right\rfloor (m - K + 2)
\geq t (K - 1) \left(\frac{m}{K - 1}\right) - \left( t(K - 1) \left(\frac{m}{K - 1}\right) - n \right) - y = n - y.
\]

The total size of \(F^*\) is \(n(K - 1) - y\), hence this is an upper bound on \(N(n, k, m, t)\).

Turning to the lower bound, by Proposition 10 there exists an optimal CBC\(^*\)(n, k, m, t) of type either \([1, K - 1]\) or \([K - 1, K]\). But if a CBC\(^*\)(n, k, m, t) belongs to the latter type and contains a member of size \(K\) as well, then its total size is greater than \(n(K - 1) - y\) and consequently it cannot be optimal. Thus, there exists an optimal CBC\(^*\)(n, k, m, t)-system \(F\) of type \([1, K - 1]\).

For every \(1 \leq i \leq K - 1\), denote by \(\ell_i\) the number of members of size \(i\) in \(F\).

The total size of \(F\) is

\[
S(F) = \sum_{i=1}^{K-1} i\ell_i = (K - 1)n - \sum_{i=1}^{K-2} (K - 1 - i)\ell_i.
\]

On the other hand, Theorem 5 yields

\[
\ell_{K-1} + \sum_{i=1}^{K-2} \ell_i \left(\frac{m - i}{K - 1 - i}\right) \leq t(K - 1) \left(\frac{m}{K - 1}\right).
\]

Substituting \(\ell_{K-1} = n - (\ell_1 + \cdots + \ell_{K-2})\), this implies

\[
\sum_{i=1}^{K-2} \ell_i \left[ \left(\frac{m - i}{K - 1 - i}\right) - 1 \right] \leq \left[ \frac{t(K - 1) \left(\frac{m}{K - 1}\right) - n}{m - K + 1} \right] = y.
\]

Now, we verify that \(S(F) \geq (K - 1)n - y\) holds. With \(p = K - 1\), Lemma 14 states that for every \(1 \leq i \leq K - 2\)

\[
K - 1 - i \leq \left[ \left(\frac{m - i}{K - 1 - i}\right) - 1 \right] \frac{m - K + 1}{m - K + 1}
\]

is valid. Together with (1) and (2) this implies

\[
S(F) = (K - 1)n - \sum_{i=1}^{K-2} (K - 1 - i)\ell_i \geq (K - 1)n - \sum_{i=1}^{K-2} \ell_i \left[ \left(\frac{m - i}{K - 1 - i}\right) - 1 \right] \frac{m - K + 1}{m - K + 1}
\geq (K - 1)n - y.
\]
Therefore, \( N(n, k, m, t) = S(\mathcal{F}) \geq (K - 1)n - y \) follows, which completes the proof of the theorem. \( \square \)

The results analogous to Theorems 13 and 15 with \( t = 1 \) were obtained in [13] and [3], respectively.

### 4. OPTIMUM VALUES FOR \( k \leq 3t \)

In this section we determine exact formulae for the minimum total size \( N(n, k, m, t) \) of combinatorial batch codes for all cases when \( k \leq 3t \) holds. Due to Observation 11, if \( \lceil k/t \rceil = 1 \) then \( N(n, k, m, t) = n \). Applying results from the previous section, formulae for the remaining cases \( t < k \leq 2t \) and \( 2t < k \leq 3t \) can be obtained.

**Theorem 16.** If \( \lceil k/t \rceil = 2 \) and \( m \geq 2 \), then

\[
N(n, k, m, t) = n \quad \text{if} \quad n \leq tm;
\]

\[
N(n, k, m, t) = 2n - tm \quad \text{if} \quad n > tm.
\]

**Proof.** Observation 11 and Theorem 13 together cover all possibilities for \( \lceil k/t \rceil = 2 \) and yield the formulae in the statement. \( \square \)

**Theorem 17.** If \( \lceil k/t \rceil = 3 \) and \( m \geq 3 \), then

\[
N(n, k, m, t) = n \quad \text{if} \quad n \leq tm;
\]

\[
N(n, k, m, t) = 2n - mt + \left\lfloor \frac{n - mt}{m - 2} \right\rfloor \quad \text{if} \quad tm < n \leq 2t \left( \frac{m}{2} \right);
\]

\[
N(n, k, m, t) = 3n - 2t \left( \frac{m}{2} \right) \quad \text{if} \quad 2t \left( \frac{m}{2} \right) < n.
\]

**Proof.** Observation 11 yields the first formula whilst Theorem 13 yields the third one, by a simple substitution. Moreover, the condition \( tm < n \leq tm(m - 1) \) corresponds to that in Theorem 15. After substituting \( \lceil k/t \rceil = 3 \), the following computation yields the second formula:

\[
N(n, k, m, t) = 2n - \left\lfloor \frac{2t \left( \frac{m}{2} \right) - n}{m - 2} \right\rfloor
\]

\[
= 2n - mt - \left\lfloor \frac{tm - n}{m - 2} \right\rfloor = 2n - mt + \left\lceil \frac{n - mt}{m - 2} \right\rceil.
\]

which concludes the proof. \( \square \)

For the particular case of \( t = 1 \) the theorems above yield a direct consequence of Theorem 8 from [13] and Theorem 1 from [3].
Acknowledgements. We thank the referees for their comments and for calling our attention to references [8, 12]. Research was supported in part by the Hungarian Scientific Research Fund, OTKA grant 81493.

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Department of Computer Science and Systems Technology, (Received August 13, 2011)
University of Pannonia, (Revised November 29, 2011)
Veszprém, Hungary
E-mail: bujtas@dcs.vein.hu
tuza@dcs.vein.hu

Computer and Automation Institute, (Received August 13, 2011)
Hungarian Academy of Sciences, (Revised November 29, 2011)
Budapest, Hungary
E-mail: tuza@sztaki.hu