ON THE METRIC DIMENSION AND FRACTIONAL METRIC DIMENSION OF THE HIERARCHICAL PRODUCT OF GRAPHS

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A set of vertices $W$ resolves a graph $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $W$. The metric dimension for $G$, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of $G$. In order to study the metric dimension for the hierarchical product $G_u^2 \oplus G_v^1$ of two rooted graphs $G_u^2$ and $G_v^1$, we first introduce a new parameter, the rooted metric dimension $\operatorname{rdim}(G_v^1)$ for a rooted graph $G_v^1$. If $G_1$ is not a path with an end-vertex $u_1$, we show that $\dim(G_u^2 \oplus G_v^1) = |V(G_2)| \cdot \operatorname{rdim}(G_v^1)$, where $|V(G_2)|$ is the order of $G_2$. If $G_1$ is a path with an end-vertex $u_1$, we obtain some tight inequalities for $\dim(G_u^2 \oplus G_v^1)$. Finally, we show that similar results hold for the fractional metric dimension.

1. INTRODUCTION

All graphs considered in this paper are nontrivial and connected. For a graph $G$, we often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For any two vertices $u$ and $v$ of $G$, denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$, and write $R_G\{u, v\} = \{w \mid w \in V(G), d_G(u, w) \neq d_G(v, w)\}$. If the graph $G$ is clear from the context, the notations $d_G(u, v)$ and $R\{u, v\}$ will be written $d(u, v)$ and $R\{u, v\}$, respectively. A subset $W$ of $V(G)$ is a resolving set of $G$ if $W \cap R\{u, v\} \neq \emptyset$ for any two distinct vertices $u$ and $v$. A metric basis of $G$ is a resolving set of $G$ with minimum cardinality. The cardinality of a metric basis of $G$ is the metric dimension for $G$, denoted by $\dim(G)$.

Metric dimension was introduced independently by Harary and Melter [15], and by Slater [24]. As a graph parameter it has numerous applications,
among them are computer science and robotics [18], network discovery and verification [5], strategies for the Mastermind game [8] and combinatorial optimization [23]. Metric dimension has been heavily studied, see [3] for a number of references on this topic.

The problem of finding the metric dimension for a graph was formulated as an integer programming problem independently by Chartrand et al. [7], and by Currie and Oellermann [10]. In graph theory, fractionalization of integer-valued graph theoretic concepts is an interesting area of research (see [22]). Currie and Oellermann [10] and Fehr et al. [11] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Arumugam and Mathew [1] initiated the study of the fractional metric dimension for graphs. For more information, see [2, 12, 13].

Let $g : V(G) \rightarrow [0,1]$ be a real function. For $W \subseteq V(G)$, denote $g(W) = \sum_{v \in W} g(v)$. The weight of $g$ is defined by $|g| = g(V(G))$. We call $g$ a resolving function of $G$ if $g(R\{u, v\}) \geq 1$ for any two distinct vertices $u$ and $v$. The minimum weight of a resolving function of $G$ is called the fractional metric dimension for $G$, denoted by $\text{dim}_f(G)$.

It was noted in [14, p.204] and [18] that determining the metric dimension for a graph is an NP-complete problem. So it is desirable to reduce the computation for the metric dimension for product graphs to the computation for some parameters of the factor graphs; see [6] for cartesian products, [16] for lexicographic products, and [25] for corona products. Recently, the fractional metric dimension for the above three products was studied in [2, 12, 13].

In order to model some real-life complex networks, Barrière et al. [4] introduced the hierarchical product of graphs and showed that it is associative. A rooted graph $G^u$ is the graph $G$ in which one vertex $u$, called root vertex, is labeled in a special way to distinguish it from other vertices. Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. The hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ is the rooted graph with the vertex set $\{x_2x_1 \mid x_i \in V(G_i), i = 1, 2\}$, having the root vertex $u_2u_1$, where $x_2x_1$ is adjacent to $y_2y_1$ whenever $x_2 = y_2$ and $\{x_1, y_1\} \in E(G_1)$, or $x_1 = y_1 = u_1$ and $\{x_2, y_2\} \in E(G_2)$. See [17, 19, 20, 21] for more information.

In this paper, we study the (fractional) metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ of rooted graphs $G_2^{u_2}$ and $G_1^{u_1}$. In Section 2, we introduce a new parameter, the rooted metric dimension $\text{rdim}(G^u)$ for a rooted graph $G^u$. If $G_1$ is not a path with an end-vertex $u_1$, we show that $\text{dim}(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \text{rdim}(G_1^{u_1})$. If $G_1$ is a path with an end-vertex $u_1$, we obtain some tight inequalities for $\text{dim}(G_2^{u_2} \sqcap G_1^{u_1})$. In Section 3, we show that similar results hold for the fractional metric dimension.

2. METRIC DIMENSION

In order to study the metric dimension for the hierarchical product of graphs, we first introduce the rooted metric dimension for a rooted graph.
A rooted resolving set of a rooted graph $G^u$ is a subset $W$ of $V(G)$ such that $W \cup \{u\}$ is a resolving set of $G$. A rooted metric basis of $G^u$ is a rooted resolving set of $G^u$ with the minimum cardinality. Here the cardinality of a rooted metric basis of $G^u$ is called rooted metric dimension of $G^u$ and denoted by $\text{rdim}(G^u)$.

The following observation is obvious.

**Observation 2.1.** If there exists a metric basis of $G$ containing $u$, then $\text{rdim}(G^u) = \text{dim}(G) - 1$. If any metric basis of $G$ does not contain $u$, then $\text{rdim}(G^u) = \text{dim}(G)$.

For graphs $H_1$ and $H_2$ we use $H_1 \cup H_2$ to denote the disjoint union of $H_1$ and $H_2$ and $H_1 + H_2$ to denote the graph obtained from the disjoint union of $H_1$ and $H_2$ by joining every vertex of $H_1$ with every vertex of $H_2$.

**Observation 2.2.** ([7]) Let $G$ be a graph of order $n$. Then $1 \leq \text{dim}(G) \leq n - 1$. Moreover,

(i) $\text{dim}(G) = 1$ if and only if $G$ is the path $P_n$ of length $n$.

(ii) $\text{dim}(G) = n - 1$ if and only if $G$ is the complete graph $K_n$ on $n$ vertices.

**Proposition 2.3.** ([7, Theorem 4]) Let $G$ be a graph of order $n \geq 4$. Then $\text{dim}(G) = n - 2$ if and only if $G = K_{s,t}$ ($s, t \geq 1$), $G = K_s + \overline{K_t}$ ($s \geq 1, t \geq 2$), or $G = K_s + (K_1 \cup K_t)$ ($s, t \geq 1$), where $\overline{K_t}$ is a null graph and $K_{s,t}$ is a complete bipartite graph.

**Proposition 2.4.** Let $G^n$ be a rooted graph of order $n$. Then $0 \leq \text{rdim}(G^n) \leq n - 2$. Moreover,

(i) $\text{rdim}(G^n) = 0$ if and only if $G = P_n$ and $u$ is one of its end-vertices.

(ii) $\text{rdim}(G^n) = n - 2$ if and only if $G = K_n$, or $G = K_{1,n-1}$ and $u$ is the centre.

**Proof.** If $G$ is a complete graph, by Observation 2.2 (ii) we have $\text{dim}(G) = n - 1$, so Observation 2.1 implies that $\text{rdim}(G^n) = n - 2$. If $G$ is not a complete graph, then $1 \leq \text{dim}(G) \leq n - 2$, which implies that $0 \leq \text{rdim}(G^n) \leq n - 2$ by Observation 2.1.

(i) Since $\text{rdim}(G^n) = 0$ if and only if $\{u\}$ is a metric basis of $G$, by Observation 2.2 (i), (i) holds.

(ii) Suppose that $\text{rdim}(G^n) = n - 2$. Then $\text{dim}(G) = n - 1$ or $n - 2$. If $\text{dim}(G) = n - 1$, then $G = K_n$. Now we consider $\text{dim}(G) = n - 2$. If $n = 3$, then $\text{dim}(G) = 1$, which implies that $G = K_{1,2}$ and $u$ is the centre. Now suppose that $n \geq 4$. Then $G$ is one of graphs listed in Proposition 2.3. If $s, t \geq 2$ or $G = K_s + (K_1 \cup K_t)$, then there exists a metric basis containing $u$, implying that $\text{rdim}(G^n) = n - 3$, which is a contradiction. Hence $G = K_{1,n-1}$. Since any metric basis of $K_{1,n-1}$ does not contain the centre, the vertex $u$ is the centre of $K_{1,n-1}$. The converse is routine.

Next, we express the metric dimension for the hierarchical product $G_2^{n_2} \cap G_1^{n_1}$ in terms of $\text{rdim}(G_1^{n_1})$.

Let $G_1^{n_1}$ and $G_2^{n_2}$ be two rooted graphs. For any two vertices $x_2x_1$ and $y_2y_1$
of $G_2^{w_2} \cap G_1^{u_1}$, observe that
\[
(1) \quad d(x_2x_1, y_2y_1) = \begin{cases} 
    d_G(x_1, y_1), & \text{if } x_2 = y_2, \\
    d_G(x_2, y_2) + d_G(x_1, u_1) + d_G(y_1, u_1), & \text{if } x_2 \neq y_2.
\end{cases}
\]

**Lemma 2.5.** Let $x_2x_1$ and $y_2y_1$ be two distinct vertices of $G_2^{w_2} \cap G_1^{u_1}$.

(i) If $x_2 = y_2$, then
\[
R\{x_2x_1, y_2y_1\} = \begin{cases} 
    \{x_2z \mid z \in R_G\{x_1, y_1\}\}, & \text{if } u_1 \notin R_G\{x_1, y_1\}, \\
    V(G_2^{w_2} \cap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_G\{x_1, y_1\}\}, & \text{if } u_1 \in R_G\{x_1, y_1\}.
\end{cases}
\]

(ii) If $x_2 \neq y_2$, then \{x_2z, y_2z\} ∩ $R\{x_2x_1, y_2y_1\} \neq \emptyset$ for any $z \in V(G_1)$.

**Proof.** (i) If $u_1 \notin R_G\{x_1, y_1\}$, then $d_G(x_1, u_1) = d_G(y_1, u_1)$. By (1), the inequality $d(x_2x_1, z_2z_1) \neq d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_G(x_1, z_1) \neq d_G(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_G\{x_1, y_1\}\}$. If $u_1 \in R_G\{x_1, y_1\}$, then $d_G(x_1, u_1) \neq d_G(y_1, u_1)$. By (1), the equality $d(x_2x_1, z_2z_1) = d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_G(x_1, z_1) = d_G(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = V(G_2^{w_2} \cap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_G\{x_1, y_1\}\}$.

(ii) Suppose that $x_2z \notin R\{x_2x_1, y_2y_1\}$. Then $d(x_2x_1, x_2z) = d(y_2y_1, x_2z)$.

By (1),
\[
d_G(x_1, z) = d_G(y_2, x_2) + d_G(y_1, u_1) + d_G(z, u_1) \geq d_G(x_2, y_2) + d_G(y_1, z),
\]
which implies that
\[
d_G(x_2, y_2) + d_G(x_1, u_1) + d_G(z, u_1) \geq 2d_G(x_2, y_2) + d_G(y_1, z) > d(y_2y_1, x_2z).
\]

Hence, $y_2z \in R\{x_2x_1, y_2y_1\}$, as desired.

**Lemma 2.6.** Let $G_1^{u_1}$ and $G_2^{w_2}$ be two rooted graphs. Then
\[
\text{rdim}(G_2^{w_2} \cap G_1^{u_1}) \geq |V(G_2)| \cdot \text{rdim}(G_1^{u_1}).
\]

**Proof.** Let $\overline{W}$ be a rooted metric basis of $G_2^{w_2} \cap G_1^{u_1}$. For $v \in V(G_2)$, write $\overline{W}_v = \{z \mid vz \in \overline{W}\}$. For any two distinct vertices $x, y$ of $G_1$, there exists a vertex $wz$ in $\overline{W} \cup \{u_2w_1\}$ such that $d(xw, wz) \neq d(uywz)$. If $w = v$, by (1) we get $d_G(x, z) \neq d_G(y, z)$, which implies that $z \in (\overline{W}_v \cup \{u_1\}) \cap R_G\{x, y\}$. If $w \neq v$, by (1) we have $d_G(x, u_1) \neq d_G(y, u_1)$, which implies that $u_1 \in R_G\{x, y\}$. Therefore, we have $\overline{W}_v \cup \{u_1\} \cap R_G\{x, y\} \neq \emptyset$, which implies that $\overline{W}_v$ is a rooted resolving set of $G_1^{u_1}$. Since $\overline{W}$ is the disjoint union of all $\overline{W}_v$’s, one gets
\[
\text{rdim}(G_2^{w_2} \cap G_1^{u_1}) = |\overline{W}| = \sum_{v \in V(G_2)} |\overline{W}_v| \geq |V(G_2)| \cdot \text{rdim}(G_1^{u_1}),
\]
as desired.

\[\square\]
Theorem 2.7. Let \( G_1^{u_1} \) and \( G_2^{u_2} \) be two rooted graphs. If \( G_1 \) is not a path with an end-vertex \( u_1 \), then
\[
\dim(G_2^{u_2} \cap G_1^{u_1}) = |V(G_2)| \cdot \text{rdim}(G_1^{u_1}).
\]

Proof. By Lemma 2.6, we only need to prove that
\[
\dim(G_2^{u_2} \cap G_1^{u_1}) \leq |V(G_2)| \cdot \text{rdim}(G_1^{u_1}).
\]
Let \( W \) be a rooted metric basis of \( G_1^{u_1} \). Then \( W \neq \emptyset \). Write \( \overline{W} = \{vw \mid v \in V(G_2), w \in W \} \). Note that \( |\overline{W}| = |V(G_2)| \cdot \text{rdim}(G_1^{u_1}) \). In order to prove (2), we only need to show that \( \overline{W} \) is a resolving set of \( G_2^{u_2} \cap G_1^{u_1} \). It suffices to show that, for any two distinct vertices \( x_2x_1 \) and \( y_2y_1 \) of \( G_2^{u_2} \cap G_1^{u_1} \),
\[
\overline{W} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset.
\]
If \( x_2 = y_2 \) and \( u_1 \notin R_{G_1}\{x_1, y_1\} \), then \( W \cap R_{G_1}\{x_1, y_1\} \neq \emptyset \), by Lemma 2.5 (i) we obtain (3). If \( x_2 = y_2 \) and \( u_1 \in R_{G_1}\{x_1, y_1\} \), by Lemma 2.5 (i) we have \( vw \in \overline{W} \cap R\{x_2x_1, y_2y_1\} \) for any \( v \neq x_2 \) and any \( w \in W \), which implies that (3) holds. If \( x_2 \neq y_2 \), since \( \{x_2w, y_2w\} \subseteq \overline{W} \) for any \( w \in W \), the inequality (3) holds by Lemma 2.5 (ii).

Combining Observation 2.1 and Theorem 2.7, we have the following result.

Corollary 2.8. Let \( G_1^{u_1} \) and \( G_2^{u_2} \) be two rooted graphs.

(i) If there exists a metric basis of \( G_1 \) containing \( u_1 \) and \( G_1 \) is not a path, then
\[
\dim(G_2^{u_2} \cap G_1^{u_1}) = |V(G_2)|(\dim(G_1) - 1).
\]

(ii) If any metric basis of \( G_1 \) does not contain \( u_1 \), then
\[
\dim(G_2^{u_2} \cap G_1^{u_1}) = |V(G_2)|\dim(G_1).
\]

The binomial tree \( T_n \) is the hierarchical product of \( n \) copies of the complete graph on two vertices, which is a useful data structure in the context of algorithm analysis and designs [9]. It was proved that the metric dimension for a tree can be expressed in terms of its parameters in [7, 15, 24].

Corollary 2.9. Let \( n \geq 2 \). Then \( \dim(T_n) = 2^{n-2} \).

Proof. Note that \( \dim(T_2) = 1 \). Now suppose \( n \geq 3 \). Since \( T_n = (K_0^0 \cap \cdots \cap K_0^0) \cap (K_2^0 \cap K_2^0) \) and \( \text{rdim}(K_0^0 \cap K_0^0) = 1 \), the desired result follows by Theorem 2.7.

We always assume that 0 is one end-vertex of \( P_n \). In the remaining of this section, we prove some tight inequalities for \( \dim(G_n \cap P_n^0) \).

Proposition 2.10. Let \( G_n \) be a rooted graph with diameter \( d \). Then
\[
\dim(G_n \cap P_n^0) \leq \dim(G_n \cap P_{n+1}^0) \text{ for } 1 \leq n \leq d - 1,
\]
\[
\dim(G_n \cap P_n^0) = \dim(G_n \cap P_{n+1}^0) \text{ for } n \geq d.
\]
Proof. If $G = K_2$, then $G^u \cap P^0_n$ is the path, which implies that (5) holds. Now we only consider $|V(G)| \geq 3$. Suppose that $\overline{W}_{n+1}$ is a metric basis of $G^u \cap P^0_{n+1}$. Let $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$. Define $\pi_n : V(G^u \cap P^0_{n+1}) \rightarrow V(G^u \cap P^0_n)$ by

$$\pi_n(v_{zi}) = \begin{cases} 
v_{zi}, & \text{if } i = n, 
v_{zi-1}, & \text{if } i \leq n - 1.
\end{cases}$$

Then $\pi_n(\overline{W}_{n+1})$ is a resolving set of $G^u \cap P^0_n$, which implies that $\dim(G^u \cap P^0_n) \leq \dim(G^u \cap P^0_{n+1})$ for any positive integer $n$. So (4) holds.

In order to prove (5), we only need to show that $\overline{W}_n$ is a resolving set of $G^u \cap P^0_{n+1}$ for $n \geq d$. Pick any two distinct vertices $v_{zi}$ and $v_{zj}$ of $G^u \cap P^0_{n+1}$. It suffices to prove that

$$\overline{W}_n \cap R_{G^u \cap P^0_{n+1}}(v_{zi}, v_{zj}) \neq \emptyset.$$  

Without loss of generality, we may assume that $0 \leq i \leq j \leq n$. If $j \leq n - 1$, then $R_{G^u \cap P^0_{n+1}}(v_{zi}, v_{zj}) \supseteq R_{G^u \cap P^0_n}(v_{zi}, v_{zj})$; and so (6) holds. Now suppose $j = n$.

Claim. There exist two distinct vertices $w_1$ and $w_2$ of $G$ such that

$$\overline{W}_n \cap \{w_{zk} \mid 0 \leq k \leq n - 1\} \neq \emptyset \text{ and } \overline{W}_n \cap \{w_{zk} \mid 0 \leq k \leq n - 1\} \neq \emptyset.$$  

Suppose for the contradiction that there exists a vertex $w \in V(G)$ such that $\overline{W}_n \subseteq \{w_{zk} \mid 0 \leq k \leq n - 1\}$. If the degree of $w$ in $G$ is one, then there exists an induced path $(w, x, y)$ in $G$. For any $w_{zk} \in \overline{W}_n$, we have $d(xz_i, wz_k) = k + 2 = d(yz_0, wz_k)$, contrary to the fact that $\overline{W}_n$ is a metric basis of $G^u \cap P^0_n$. If the degree of $w$ in $G$ is at least two, pick two distinct neighbors $x$ and $y$ of $w$ in $G$. Then $d(xz_i, wz_k) = k + 1 = d(yz_0, wz_k)$ for any $w_{zk} \in \overline{W}_n$, a contradiction. Hence our claim is valid.

Now we prove (6) for $j = n$. By the claim, we may pick two distinct vertices $w_1$ and $w_2$ satisfying (7).

Case 1. $v_1 = v_2$. Since $\{w_{zk} \mid 0 \leq k \leq n - 1\}$ or $\{w_{zk} \mid 0 \leq k \leq n - 1\}$ is a subset of $R_{G^u \cap P^0_{n+1}}(v_{zi}, v_{zn})$, the inequality (6) holds.

Case 2. $v_1 \neq v_2$.

Case 2.1. $i = 0$. Without loss of generality, we may assume that $w_1 \neq v_2$. Pick $z_k$ satisfying $w_{zk} \in \overline{W}_n$. Then $d(v_{z0}, w_{zk}) = d_G(v_1, v_1) + k \leq d + k \leq n + k < d_G(v_2, v_1) + n + k = d(v_{zn}, w_{zk})$, which implies that $w_{zk} \in R_{G^u \cap P^0_{n+1}}(v_{z0}, v_{zn})$. So (6) holds.

Case 2.2. $i \geq 1$. Note that

$$R_{G^u \cap P^0_{n+1}}(v_{zi}, v_{zn}) = R_{G^u \cap P^0_{n+1}}(v_{zi-1}, v_{zn-1}) \supseteq R_{G^u \cap P^0_n}(v_{zi-1}, v_{zn-1}).$$

Then (6) holds.  \(\square\)
For any rooted graph $G^n$, we have
\begin{equation}
\dim(G) \leq \dim(G^n \cap P_n^0) \leq |V(G)| - 1.
\end{equation}

**Proof.** Let $z$ be the other end-vertex of $P_n$. Fix a vertex $v_0 \in V(G)$ and write $S = \{ vz | v \in V(G) \setminus \{ v_0 \} \}$. Since $\{ z \}$ is a resolving set of $P_n$, the set $S$ resolves $G \cap P_n$ by (1). Hence $\dim(G^n \cap P_n^0) \leq |S| = |V(G)| - 1$. Since $G^n$ is isomorphic to $G^n \cap P_1^0$, Proposition 2.10 implies that $\dim(G) \leq \dim(G^n \cap P_n^0)$. \hfill\Box

For $m \geq 2$, we have $\dim(K_m^n \cap P_n^0) = m - 1$. This shows that the inequalities (4) and (8) are tight.

**Example 2.12.** For $m \geq 3$ and $n \geq 2$, we have $\dim(P_m^n \cap P_0^0) = 2$. In fact, write $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$, then $\{ z_0 z_{n-1}, z_{m-1} z_{n-1} \}$ is a resolving set of $P_m^n \cap P_0^0$.

**Example 2.13.** Let $C_m$ be the cycle with length $m$. Then $\dim(C_m^n \cap P_0^0) = 2$. In fact, write $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$ and $C_m = (c_0, c_1, \ldots, c_{m-1}, c_0)$, then $\{ c_0 z_{n-1}, c_1 z_{n-1} \}$ is a resolving set of $C_m^n \cap P_0^0$.

3. FRACTIONAL METRIC DIMENSION

In order to study the fractional metric dimension for the hierarchical product of graphs, we first introduce the fractional rooted metric dimension for a rooted graph.

Similar to the fractionalization of metric dimension, we give a fractional version of the rooted metric dimension for a rooted graph. Let $G^n$ be a rooted graph of order $n$. Write
\[
\mathcal{P}^n = \{ \{ v, w \} \mid v, w \in V(G), v \neq w, d(v, u) = d(w, u) \}.
\]

Suppose that $\mathcal{P}^n \neq \emptyset$. Write $V(G) \setminus \{ u \} = \{ v_1, \ldots, v_{n-1} \}$ and $\mathcal{P}^n = \{ \alpha_1, \ldots, \alpha_m \}$. Let $A^n$ be the $m \times (n - 1)$ matrix with
\[
(A^n)_{ij} = \begin{cases} 
1, & \text{if } v_j \text{ resolves } \alpha_i, \\
0, & \text{otherwise}.
\end{cases}
\]

The integer programming formulation of the rooted metric dimension for $G^n$ is given by
\[
\begin{align*}
\text{Minimize } f(x_1, \ldots, x_{n-1}) &= x_1 + \cdots + x_{n-1} \\
\text{Subject to } A^n x &\geq 1
\end{align*}
\]
where $x = (x_1, \ldots, x_{n-1})^T$, $x_i \in \{0, 1\}$ and $1$ is the $m \times 1$ column vector all of whose entries are $1$. The optimal solution of the linear programming relaxation of the above integer programming problem, where we replace $x_i \in \{0, 1\}$ by $x_i \in [0, 1]$, gives the fractional rooted metric dimension for $G^n$, which we denote by $\text{rdim}_f(G^n)$.

Let $G^n$ be a rooted graph which is not a path with an end-vertex $u$. A rooted resolving function of a rooted graph $G^n$ is a real value function $q : V(G) \rightarrow [0, 1]$ such that $q(R(v, w)) \geq 1$ for each $\{ v, w \} \in \mathcal{P}^n$. The fractional rooted metric dimension for $G^n$ is the minimum weight of a rooted resolving function of $G^n$. 
Proposition 3.1. Let $G^n$ be a rooted graph which is not a path with an end-vertex $u$. Then

(i) $\text{rdim}_f(G^n) \leq \text{rdim}(G^n)$.

(ii) $\text{rdim}_f(G^n) \leq \frac{|V(G)| - 1}{2}$.

(iii) $\dim_f(G) - 1 \leq \text{rdim}_f(G^n) \leq \text{dim}_f(G)$.

Proof. (i) Let $W$ be a rooted metric basis of $G^n$. Define $g : V(G) \rightarrow [0, 1]$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in W, \\ 0, & \text{if } v \notin W. \end{cases}$$

For any $\{x, y\} \in P^n$, there exists a vertex $v \in W$ such that $d(x, v) \neq d(y, v)$. Then $g(R(x, y)) \geq g(v) = 1$, which implies that $g$ is a rooted resolving function of $G^n$. Hence $\text{rdim}_f(G^n) \leq |g| = |W| = \text{rdim}(G^n)$.

(ii) The function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(v) = \begin{cases} 0, & \text{if } v = u, \\ \frac{1}{2}, & \text{if } v \neq u \end{cases}$$

is a rooted resolving function of $G^n$. Hence $\text{rdim}_f(G^n) \leq \frac{|V(G)| - 1}{2}$.

(iii) It is clear that $\text{rdim}_f(G^n) \leq \text{dim}_f(G)$. Let $g$ be a rooted resolving function of $G^n$. Then the function $h : V(G) \rightarrow [0, 1]$ defined by

$$h(v) = \begin{cases} 1, & \text{if } v = u, \\ g(v), & \text{if } v \neq u \end{cases}$$

is a resolving function of $G$. Hence $\dim_f(G) \leq \text{rdim}_f(G^n) + 1$, as desired. \hfill $\Box$

If $u$ is not an end-vertex of the path $P_n$, then $\text{rdim}_f(P^n) = \text{rdim}(P^n) = \dim_f(P_n) = 1$, which implies that the upper bounds in Proposition 3.1 (i) and (iii) are tight. The fact that $\text{rdim}_f(K^n_u) = \frac{n - 1}{2}$ shows that the inequality in Proposition 3.1 (ii) is tight.

Next, we study the fractional metric dimension for the hierarchical product of graphs.

For two rooted graphs $G_1^n$ and $G_2^n$, write

$P^n_1 = \{\{x, y\} \subseteq V(G_1) | x \neq y, d_{G_1}(x, u_1) = d_{G_1}(y, u_1)\}$,

$P_{2u_1} = \{\{x_{2x_1}, y_{2y_1}\} \subseteq V(G_2^n) | x_{2x_1} \neq y_{2y_1}, d(x_{2x_1}, u_2u_1) = d(y_{2y_1}, u_2u_1)\}$.

Lemma 3.2. Let $G_1^n$ and $G_2^n$ be two rooted graphs. If $G_1$ is not a path with an end-vertex $u_1$, then

$$\text{rdim}_f(G_2^n \cap G_1^n) \geq |V(G_4)| \cdot \text{rdim}_f(G_1^n).$$
Proof. Suppose that \( \overline{f} \) is a rooted resolving function of \( G_2^{u_2} \cap G_1^{u_1} \) with weight \( \text{rdim}_f(G_2^{u_2} \cap G_1^{u_1}) \). For each \( z \in V(G_2) \), define
\[
\overline{f}_z : V(G_1) \rightarrow [0, 1], \quad x \mapsto \overline{f}(zx).
\]
Write \( \overline{P}^{u_1} = \{ \{zx, zy\} \mid z \in V(G_2), \{x, y\} \in P^{u_1}\} \). By (1), we have \( \overline{P}^{u_1} \subseteq \overline{P}^{u_2 \cap u_1} \). Hence \( \overline{f}_z(R_{G_1}\{x, y\}) \geq 1 \) for any \( \{x, y\} \in P^{u_1} \), which implies that \( |\overline{f}_z| \geq \text{rdim}_f(G_1^{u_1}) \). Consequently,
\[
\text{rdim}_f(G_2^{u_2} \cap G_1^{u_1}) = |\overline{f}| = \sum_{z \in V(G_2)} |\overline{f}_z| \geq |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}),
\]
as desired.

Theorem 3.3. Let \( G_1^{u_1} \) and \( G_2^{u_2} \) be two rooted graphs. If \( G_1 \) is not a path with an end-vertex \( u_1 \), then
\[
\dim_f(G_2^{u_2} \cap G_1^{u_1}) = |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}).
\]

Proof. Combining Proposition 3.1 and Lemma 3.2, we only need to prove that
\[
\text{dim}_f(G_2^{u_2} \cap G_1^{u_1}) \leq |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}).
\]
By Proposition 2.4 we have \( P^{u_1} \neq \emptyset \). Let \( g \) be a rooted resolving function of \( G_1 \) with weight \( \text{rdim}_f(G_1^{u_1}) \). Define
\[
\overline{f} : (G_2^{u_2} \cap G_1^{u_1}) \rightarrow [0, 1], \quad x_2 x_1 \mapsto g(x_1).
\]
We shall show that, for any two distinct vertices \( x_2 x_1 \) and \( y_2 y_1 \) of \( G_2^{u_2} \cap G_1^{u_1} \),
\[
\overline{f}(R\{x_2 x_1, y_2 y_1\}) \geq 1.
\]

Case 1. \( x_2 = y_2 \). If \( u_1 \not\in R_{G_1}\{x_1, y_1\} \), by Lemma 2.5 we get \( R\{x_2 x_1, y_2 y_1\} = \{x_2 z \mid z \in R_{G_1}\{x_1, y_1\}\} \), which implies that \( \overline{f}(R\{x_2 x_1, y_2 y_1\}) = g(R_{G_1}\{x_1, y_1\}) \). Since \( \{x_1, y_1\} \in P^{u_1} \), we obtain (10). If \( u_1 \in R_{G_1}\{x_1, y_1\} \), by Lemma 2.5 we have \( R\{x_2 x_1, y_2 y_1\} \supseteq \{vx \mid v \in V(G_1)\} \) for any \( v \in V(G_2) \setminus \{x_2\} \), which implies that \( \overline{f}(R\{x_2 x_1, y_2 y_1\}) \geq |g| \), so (10) holds.

Case 2. \( x_2 \neq y_2 \). Write \( W = \{z \mid x_2 z \in R\{x_2 x_1, y_2 y_1\}\} \) and \( S = \{z \mid y_2 z \in R\{x_2 x_1, y_2 y_1\}\} \). By Lemma 2.5 we have \( W \cup S = V(G_1) \). Then
\[
\overline{f}(R\{x_2 x_1, y_2 y_1\}) \geq \sum_{z \in W} \overline{f}(x_2 z) + \sum_{z \in S} g(W) + g(S) \overline{f}(y_2 z) \geq |g|,
\]
which implies that (10) holds.

Therefore, \( \overline{f} \) is a resolving function of \( G_2^{u_2} \cap G_1^{u_1} \), which implies that \( \dim_f(G_2^{u_2} \cap G_1^{u_1}) \leq |\overline{f}| \). Since \( |\overline{f}| = |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}) \), we obtain (9). Our proof is accomplished.

By Theorem 3.3, we obtain the following corollary immediately.
Corollary 3.4. Let \( n \geq 2 \). Then \( \dim_f(T_n) = 2^{n-2} \).

Arumugam and Mathew [1] proposed a natural problem: Characterize graphs for which \( \dim_f(G) = \dim(G) \). By Corollaries 2.9 and 3.4, the binomial tree \( T_n \) satisfies \( \dim_f(T_n) = \dim(T_n) \). But this problem is still open.

It seems that there is a gap to determine \( \dim_f(G^u \sqcap P_n^0) \). We conclude this paper by giving some tight inequalities involving it.

Proposition 3.5. For any rooted graph \( G^u \), we have

\[
\dim_f(G) \leq \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2}.
\]

Proof. Write \( P_n = (z_0 = 0, z_1, \ldots, z_{n-1}) \). For a resolving function \( \mathcal{F}_{n+1} \) of \( G^u \sqcap P_{n+1}^0 \), we define \( \mathcal{F}_{n+1} : V(G^u \sqcap P_n^0) \rightarrow [0, 1] \) by

\[
\mathcal{F}_{n+1}(x_2 x_1) = \begin{cases} 
\mathcal{F}_{n+1}(x_2 z_{n-1}) + \mathcal{F}_{n+1}(x_2 z_n), & \text{if } x_1 = z_{n-1}, \\
\mathcal{F}_{n+1}(x_2 x_1), & \text{if } x_1 \neq z_{n-1}.
\end{cases}
\]

Then \( \mathcal{F}_{n+1} \) is a resolving function of \( G^u \sqcap P_n^0 \). Since \( |\mathcal{F}_{n+1}| = |\mathcal{F}_{n+1}| \), we have

\[
\dim_f(G) = \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0) \leq \dim_f(G^u \sqcap P_{n+1}^0).
\]

For proving the last inequality, define \( \overline{h} : V(G^u \sqcap P_{n+1}^0) \rightarrow [0, 1] \) by

\[
\overline{h}(x_2 x_1) = \begin{cases} 
\frac{1}{2}, & \text{if } x_1 = z_n, \\
0, & \text{if } x_1 \neq z_n.
\end{cases}
\]

Then \( \overline{h} \) is a resolving function of \( G^u \sqcap P_{n+1}^0 \) with weight \( \frac{|V(G)|}{2} \). Hence \( \dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2} \). \( \Box \)

For \( m \geq 2 \), we have \( \dim_f(K_m^u \sqcap P_n^0) = \frac{m}{2} \). This shows that all the inequalities in Proposition 3.5 are tight.

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