MULTIPLE POSITIVE SOLUTIONS FOR DISCRETE $p$-LAPLACIAN EQUATIONS WITH POTENTIAL TERM

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We study the existence of solutions to nonlinear discrete boundary value problems with the discrete $p$-Laplacian, potential, and nonlinear source terms. Using variational methods, we demonstrate that there exist at least two positive solutions. The existence strongly depends on the smallest positive eigenvalue of Dirichlet eigenvalue problems and the growth conditions of the source terms.

1. INTRODUCTION

Discrete boundary value problem is one of the most important mathematical equations and has rich applications in the area such as astrophysics, gas dynamics, fluid mechanics, computer science, image processing, chemically reacting systems, and others. Study of the properties of the operators plays a key role in dealing with these problems. Recently the discrete $p$-Laplacian, which appears in various discrete problems, has received great attention from many researchers. For more details, see [3, 4, 5, 11, 12, 13].

In [1], Agarwal, Perera and O’Regan proved the existence of multiple positive solutions to the following boundary value problem involving the discrete $p$-Laplacian:

\[
\begin{aligned}
-\mathcal{D}(\phi_p(Du(k-1))) &= f(k,u(k)), \quad k \in [1,T] := \{1,\ldots,T\} \\
u(0) &= u(T+1) = 0.
\end{aligned}
\]
where $T$ is a fixed positive integer, $Du(k) := u(k+1) - u(k)$ is the forward difference operator, $\phi_p(t) := |t|^{p-2}t$, $t \in \mathbb{R}$, $1 < p < \infty$, and the function $f \in C([1, T] \times (0, \infty); \mathbb{R})$ satisfies

$$
(1.2) \quad a_0(k) \leq f(k, t) \leq a_1(k)t^{-\gamma}, \quad (k, t) \in [1, T] \times (0, t_0)
$$

for some nontrivial functions $a_0, a_1 \geq 0$ and $\gamma, t_0 > 0$. Their first result is that if the function $f$ satisfies (1.2) and

$$
(1.3) \quad \limsup_{t \to \infty} \frac{f(k, t)}{t^{p-1}} < \lambda_1, \quad k \in [1, T],
$$

where $\lambda_1$ is the smallest positive eigenvalue of

$$
\begin{cases}
-\mathcal{D}(\phi_p(Du(k-1))) = \lambda\phi_p(u(k)), & k \in [1, T] \\
u(0) = u(T + 1) = 0
\end{cases}
$$

then (1.1) has a solution. Their second result is that if $f$ satisfies (1.2) and

$$
(1.4) \quad f(k, t_1) \leq 0, \quad k \in [1, T],
$$

for some $t_1 > t_0$, then (1.1) has a solution $u_1 < t_1$. If, in addition, $f$ satisfies

$$
(1.5) \quad \liminf_{t \to \infty} \frac{f(k, t)}{t^{p-1}} > \lambda_1, \quad k \in [1, T],
$$

then there exists a second solution $u_1 < u_2$.

The main purpose of this paper is to generalize the graph structure and the main equation and improve the growth conditions in [1]. To do this, we consider a discrete boundary value problem including potential terms on a graph. Namely, we deal with the following equation on a simple and connected graph $G = G(S \cup \partial S, E)$:

$$
(1.6) \quad \begin{cases} 
-\Delta_{p, \omega}u(x) + V(x)|u(x)|^{p-2}u(x) = f(x, u(x)), & x \in S \\
u(x) = \sigma(x) \geq 0, & x \in \partial S
\end{cases}
$$

where $\Delta_{p, \omega}$ is the discrete $p$-Laplacian defined by

$$
\Delta_{p, \omega}u(x) := \sum_{y \in S} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y), \quad x \in S
$$

and $\omega : S \cup \partial S \times S \cup \partial S \to [0, \infty)$ is a weight function defined by

(i) $\omega(x, y) = \omega(y, x) > 0$ if $\{x, y\} \in E$,

(ii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin E$.

We note that the operator $\mathcal{D}(\phi_p(Du))$ in (1.1) is the discrete $p$-Laplacian $\Delta_{p, \omega}$ on a path with standard weights.

We now propose the following assumptions:
Multiple positive solutions for discrete $p$-Laplacian equations

(H1) $V$ satisfies $\lambda_{1,V} > 0$.

(H2) $f$ satisfies

$$a_0(x) \leq f(x,t), \quad t \in (0,\tau_0(x)), \quad x \in S,$$

where $\tau_0$ is a positive function on $S$ and $a_0$ is a non-negative function satisfying $a_0(x) \neq 0$ for some $x \in S$.

(H3) $f$ satisfies

$$\limsup_{t \to \infty} \frac{f(x,t)}{t^{p-1}} < \lambda_{1,V}, \quad x \in S.$$

(H3)' $\frac{1.6}{(1.6)}$ has a supersolution $w_0$.

(H4) $f$ satisfies

$$\liminf_{t \to \infty} \frac{f(x,t)}{t^{p-1}} > \lambda_{1,V}, \quad x \in S.$$

The assumptions (H3), (H3)', and (H4) provide more improved bounds than (1.3), (1.4) and (1.5). Main results in this paper are as follows:

**Theorem 1.** Let (H1), (H2), and (H3) hold. Then (1.6) has a positive solution $u$.

**Theorem 2.** If (H1), (H2), and (H3)' hold, then (1.6) has a positive solution $u_1$ satisfying $u_1 < w_0$. If, in addition, (H4) holds, then (1.6) has the second positive solutions $u_2$ satisfying $u_1 < u_2$.

The rest of this paper is organized as follows: in Section 2, we present graph theoretic notions used frequently throughout this paper. We also introduce a comparison principle and the sub-supersolution method for the discrete $p$-Laplacian with potential terms which are proved in [7, 8]. In Section 3, we show the existence of a positive solution to our problem. In Section 4, we prove that there exist at least two positive solutions, and verify that one of them is strictly greater than the other. Finally, in Section 5, we give some examples for the results in Section 4.

**2. PRELIMINARIES**

In this section, we start with graph theoretic notions used frequently throughout this paper.

By a graph $G = G(S \cup \partial S, E)$ we mean a two finite and disjoint set $S$ and $\partial S$ of vertices, called interior and boundary respectively, with a set $E$ of unordered pairs of distinct elements of $S \cup \partial S$ whose elements are called edges. As conventionally used, we denote by $x \in \overline{S}$ the facts that $x$ is a vertex in $S \cup \partial S$.

A graph $G$ is said to be simple if it has neither multiple edges nor loops, and $G$ is said to be connected if for every pair of vertices $x$ and $y$, there exists a sequence (termed a path) of vertices $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ such that $x_j$ and $x_j$ are connected by an edge (termed adjacent) for $j = 1, \ldots, n$. 
A graph $G' = G'(T, E')$ is said to be a subgraph of $G = G(S, E)$. If $T \subset S$ and $E' \subset E$. If $E'$ consists of all the edges from $E$ which connect the vertices $T$ in a graph $G$ then $G'$ is called an induced subgraph.

A weight on a graph $G$ is a function $\omega : S \times S \rightarrow [0, \infty)$ satisfying

(i) $\omega(x, y) = \omega(y, x) > 0$ if $\{x, y\} \in E$,

(ii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin E$.

A graph associated with a weight is said to be a weighted graph (or network).

Throughout this paper, a function on a graph is understood as a function defined on the set of vertices of the graph. For a nonempty subset $T$ of vertices in $G$, the integration of a function $u : T \rightarrow \mathbb{R}$ is defined by

$$\int_T u := \sum_{x \in T} u(x).$$

For $p > 1$, the $p$-directional derivative of a function $u : S \rightarrow \mathbb{R}$ in the direction $y$ is defined by

$$D_{p, \omega, y} u(x) := |u(y) - u(x)|^{p-2}(u(y) - u(x))\sqrt{\omega(x, y)}$$

for $x \in S$. The $p$-gradient $\nabla_{p, \omega}$ of a function $u : S \rightarrow \mathbb{R}$ is defined to be

$$\nabla_{p, \omega} u(x) := (D_{p, \omega, y} u(x))_{y \in S}$$

for $x \in S$. In the case of $p = 2$, we write simply $\nabla_{\omega}$ instead of $\nabla_{2, \omega}$.

The discrete $p$-Laplacian $\Delta_{p, \omega}$ of a function $u : S \rightarrow \mathbb{R}$ is defined by

$$\Delta_{p, \omega} u(x) := \sum_{y \in S} |u(y) - u(x)|^{p-2}(u(y) - u(x))\omega(x, y), \quad x \in S.$$

We note that for any pair of functions $u : S \rightarrow \mathbb{R}$ and $v : S \rightarrow \mathbb{R}$, we have

$$2\int_S v(-\Delta_{p, \omega} u) = \int_S \nabla_{\omega} v \cdot \nabla_{p, \omega} u.$$ (2.7)

where $A \cdot B := \sum_{i=1}^n a_i b_i$ for $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. We remark here that other authors define the $p$-Laplacian as generalizations of the combinatorial graph Laplacian which then has opposite sign, see e.g. [2]. In this paper we follow the notions in [1].

In this paper, we define a set $A_\sigma$ for a function $\sigma : \partial S \rightarrow [0, \infty)$ as follows:

$$A_\sigma := \{u : S \rightarrow \mathbb{R} \mid u(z) = \sigma(z), z \in \partial S\}.$$
In particular, in the case of $\sigma \equiv 0$, we write $A_0$.

For a function $V : S \to \mathbb{R}$, Dirichlet eigenvalue problem for the discrete $p$-Laplacian with potential term is defined as follows:

$$\begin{cases}
-\Delta_p \phi(x) + V(x)\phi(x)|^{p-2}\phi(x) = \lambda |\phi(x)|^{p-2}\phi(x), & x \in S \\
\phi(z) = 0,
\end{cases}$$

This problem has the first eigenvalue which is given by

$$\lambda_{1,V} := \inf_{\phi \in A_0} \frac{\frac{1}{2} \int_S \nabla \phi \cdot \nabla_p \phi + \int_S V|\phi|^p}{\int_S |\phi|^p}$$

and there exists a positive eigenfunction $\phi_1 \in A_0$ corresponding to $\lambda_{1,V}$ satisfying

$$\int_S |\phi_1|^p = 1.$$

We note that the first eigenvalue $\lambda_{1,V}$ can be considered as a functional with respect to $V$ and it has the following properties:

(i) $\lambda_{1,V}$ is continuous on $A_0$,
(ii) the multiplicity of $\lambda_{1,V}$ is one,
(iii) $\lambda_{1,V}$ is isolated.

Particularly, if we put $V \equiv 0$ and $\partial S \neq \emptyset$, then $\lambda_{1,V} > 0$ (for more details, see [9]).

We include here a (strong) comparison principle and the method of sub-supersolutions for the discrete $p$-Laplacian for future use which are proved in [7, 8].

**Theorem 2.1 (Comparison principle).** For a function $V : S \to \mathbb{R}$, suppose that $\lambda_{1,V} > 0$ and $u_i : S \to \mathbb{R}$, $i = 1, 2$ satisfies

$$\begin{cases}
-\Delta_p u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \geq \lambda_{1,V} |u_2(x)|^{p-2}u_2(x), & x \in S \\
u_2(x) \geq u_1(x), & x \in \partial S.
\end{cases}$$

If we assume in addition that

$$\begin{cases}
-\Delta_p u_2(x) + V(x)|u_2(x)|^{p-2}u_2(x) \geq 0, & x \in S \\
u_2(x) \geq 0, & x \in \partial S
\end{cases}$$

then $u_2(x) \geq u_1(x)$ for all $x \in S$. Moreover, the equalities hold in (2.10) if and only if $u_2 \equiv u_1$.

**Theorem 2.2 (Sub-supersolution method).** For a function $f \in C(S \times \mathbb{R}; \mathbb{R})$, suppose that $\underline{u}$ and $\overline{u}$ in $A_{\sigma}$ are subsolution and supersolution with $\underline{u} \leq \overline{u}$ to the equation

$$\begin{cases}
-\Delta_p u(x) = f(x, u(x)), & x \in S \\
u(x) = \sigma(x), & x \in \partial S.
\end{cases}$$

If a given function $f$ satisfies that there exists $\lambda > 0$ such that $f(\cdot, t) + \lambda |t|^{p-2}t$ is nondecreasing in $S$, then there exists a solution $u$ of (2.11) such that $\underline{u} \leq u \leq \overline{u}$. 
We note that for a non-zero function $a_0 : S \to [0, \infty)$, the equation
\begin{equation}
\begin{aligned}
-\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = a_0(x), & \quad x \in S \\
u(x) = \sigma(x) \geq 0, & \quad x \in \partial S
\end{aligned}
\end{equation}
has a unique solution $u_0$ if the first eigenvalue $\lambda_{1,V}$ is bigger than zero. Moreover, the condition $a_0 \not\equiv 0$ implies that the solution $u_0$ is strictly positive on $S$ (for details, see [8]).

### 3. A POSITIVE SOLUTION

For a function $g \in C(S \times \mathbb{R}; \mathbb{R})$, we consider a functional $E_g$ defined by
\[ E_g[u] := \frac{1}{2p} \int_S \nabla_{p,\omega} u \cdot \nabla_{\omega} u + \frac{1}{p} \int_S V|u|^p - \int_S G_u, \quad u \in \mathcal{A}_\sigma \]
where $G_u : S \to \mathbb{R}$ is defined by
\[ G_u(x) := \int_0^{u(x)} g(x, t) \, dt. \]
We note that since the functional $E_g$ is differentiable, a critical point of $E_g$ is a solution to (1.6).

**Lemma 3.3.** Suppose that a function $g \in C(S \times \mathbb{R}; \mathbb{R})$ satisfies
\begin{equation}
\limsup_{|t| \to \infty} \frac{g(x, t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S.
\end{equation}
Then there exists a solution to
\begin{equation}
\begin{aligned}
-\Delta_{p,\omega} u(x) + V(x)|u(x)|^{p-2}u(x) = g(x, u(x)), & \quad x \in S \\
u(x) = \sigma(x) \geq 0, & \quad x \in \partial S.
\end{aligned}
\end{equation}

**Proof.** It follows from (3.13) that there exists $M_0 < \lambda_{1,V}$ such that
\begin{equation}
\int_0^t g(x, s) \, ds \leq \frac{M_0}{p} |t|^p + C, \quad x \in S, \ t \in \mathbb{R}
\end{equation}
for some constant $C$. Since $\lambda_{1,V}$ is continuous with respect to $V$, there exists a sufficiently small value $\epsilon > 0$ such that $M_0 < (1 - \epsilon)\lambda_{1,V}$. We now define two functions $u_t$ and $\bar{u}$ by
\[ u_t(x) := \begin{cases} 
  tu(x), & x \in S \\
  \sigma(x), & x \in \partial S
\end{cases}, \quad x \in \mathbb{R} \]
and
\[ \bar{u}(x) := \begin{cases} 
  u(x), & x \in S \\
  0, & x \in \partial S
\end{cases}, \quad x \in \mathbb{R}. \]
for \( t > 0 \) and \( u \in \mathcal{A}_\sigma \) satisfying \( \int_S |u|^p = 1 \). Then for each \( t \) and \( u \in \mathcal{A}_\sigma \) with \( \int_S |u|^p = 1 \), we have

\[
E_g[u_t] \geq \frac{\|p\}}{2p} \sum_{x,y \in S} |y(x) - u(x)|^p \omega(x, y) + \frac{1}{p} \sum_{y \in \partial S} |\sigma(y) - tu(x)|^p \omega(x, y)
+ \frac{1}{p} \sum_{x \in S} \int \omega(x, y) + \frac{1}{p} \sum_{x,y \in \partial S} |\sigma(y) - \sigma(x)|^p \omega(x, y)
- M_0p \sum_{x \in S} |u(x)|^p - C'.
\]

We note that for \( |x| > \gamma > 0, \beta \geq 0 \), there exists \( K > 0 \) such that

\[
(3.15) \quad |\alpha t - \beta|^p \geq \gamma t^p - K \beta, \quad t > 0.
\]

It follows from (3.15) that there exists \( K > 0 \) such that

\[
|\sigma(y) - tu(x)|^p = |u(x) t - \sigma(y)|^p \geq (1 - \epsilon)|u(x)|^p t^p - K \sigma(y)
\]

for all \( x \in S \) and \( y \in \partial S \). Thus we have

\[
\sum_{x \in S} \sum_{y \in \partial S} |\sigma(y) - tu(x)|^p \omega(x, y) \geq t^p (1 - \epsilon) \sum_{x \in S} |u(x)|^p \omega(x, y) - K \sum_{y \in \partial S} \sigma(y) \omega(x, y)
\]

\[
= t^p (1 - \epsilon) \sum_{x \in S} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x, y) - K \sum_{y \in \partial S} \sigma(y) \omega(x, y).
\]

Therefore the functional \( E_g \) satisfies that

\[
E_g[u_t] \geq \frac{(1 - \epsilon) t^p}{2p} \sum_{x,y \in S} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x, y) + \frac{(1 - \epsilon) t^p}{p} \sum_{x \in S} \int \omega(x, y) + \frac{1}{p} \sum_{x \in S} V(x) |\tilde{u}(x)|^p
+ \frac{1}{2p} \sum_{x,y \in \partial S} |\sigma(y) - \sigma(x)|^p \omega(x, y) - M_0p \sum_{x \in S} |u(x)|^p - C'.
\]

By the definition of \( \lambda_1 \), it holds that

\[
\frac{1}{2} \sum_{x,y \in S} |\tilde{u}(y) - \tilde{u}(x)|^p \omega(x, y) + \sum_{x \in S} \int \omega(x, y) \geq \lambda_1 \sum_{x \in S} \tilde{u}(x).
\]
Thus it follows from \( \int_S |u|^p = 1 \) that
\[
E_g[u_t] \geq \frac{1}{p} (1 - \epsilon) \lambda_1 (1 - M_0) t^p + \frac{1}{2p} \sum_{x,y \in \partial S} |\sigma(y) - \sigma(x)|^p \omega(x,y)
\]
\[
- K \sum_{x,y \in S} \sigma(y) \omega(x,y) - C'
\]
\[
\rightarrow \infty, \quad t \to \infty.
\]

Thus \( E_g \) has a global minimizer which is a solution to (1.6).

Remark 3.4. In [1] the special case of \( V \equiv 0 \) and \( \sigma \equiv 0 \) was shown.

Example 1. Let a graph \( G \) be given by a path, \( V \) be nonnegative and non-zero on \( S \), and \( \sigma \equiv 0 \). Then by the definition of the first eigenvalue, it holds that \( \lambda_{1,V} > \lambda_{1,0} \).

We now putting \( h(x,t) := g(x,t) - V(x) |t|^{p-2} t \), then \( h \) satisfies
\[
\limsup_{|t| \to \infty} \frac{h(x_0,t)}{|t|^{p-2} t} \geq \lambda_{1,0}
\]
and it holds that
\[
-\Delta_{p,\omega} u(x) = h(x,u(x)), \quad x \in S
\]
\[
u(x) = 0, \quad x \in \partial S.
\]

In this case, the function \( h \) does not satisfy the hypothesis in [1, Lemma 2.4] but by Lemma 3.3, the equation has a solution.

We now prove the first main result in this paper.

**Theorem 3.5.** Suppose \( (H1), (H2), \) and \( (H3) \) hold. Then (1.6) has a positive solution \( u \) satisfying
\[
\epsilon_0^{p-1} u_0(x) < u(x), \quad x \in S
\]
where \( u_0 \) is a strictly positive solution to (2.12) and
\[
\epsilon_0 := \min \left\{ \min_{x \in S} \left( \frac{\tau_0(x)}{u_0(x)} \right)^{p-1}, 1 \right\}.
\]

**Proof.** We first define a function \( \underline{u} \) by \( \underline{u}(x) := \epsilon_0^{p-1} u_0(x) \) for all \( x \in S \). Then it is clear that \( \underline{u}(x) \leq \tau_0(x), \quad x \in S \). We now consider a function \( f: S \to \mathbb{R} \) defined by
\[
f(x,t) := \begin{cases} f(x, \underline{u}(x)), & t \leq \underline{u}(x) \\ f(x,t), & t > \underline{u}(x) \end{cases}
\]
for $x \in S$. Then the function $f$ satisfies
\[
\limsup_{|t| \to \infty} \frac{f(x, t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S.
\]
Hence by Lemma 3.3, there exists a global minimizer $u_1$ of $E_f$. It is a solution to the equation
\[
\begin{cases}
-\Delta_{p,w} u(x) + V(x)|u(x)|^{p-2}u(x) = f(x, u(x)), & x \in S \\
u(x) = \sigma(x), & x \in \partial S.
\end{cases}
\]
We now show that $\lambda \leq \phi$ where \( \phi \in x \) for all $x \in S$. Since $f(0) < \lambda$, it follows from Theorem 3.5 that there exists a positive solution to (1.6). On the other hand, it follows from the assumptions of $G$ that an induced subgraph $G(S, E)$ is connected and that $u_1(x) \neq u(x)$, $x \in \partial T := \{x \in S \setminus T \mid \exists \gamma \in T\}$. Since $u_1(x) \leq u(x)$ for $x \in T$, by the definition of $f$, we have
\[
-\Delta_{p,w} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) = f(x, u_1(x)) = f(x, u(x)), \quad x \in T.
\]
Since $f(x, u(x)) \geq a_0(x) \geq c_0 a_0(x)$, $x \in T$, it follows that
\[
-\Delta_{p,w} u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) \geq -\Delta_{p,w} u_1(x) + V(x)|u(x)|^{p-2}u(x)
\]
for all $x \in T$. Since $u_1(x) > u(x)$ for all $x \in \partial T$, by the comparison principle, the set $T$ is empty. Hence $u_1 > u$ on $S$.

**Example 2.** Let a graph $G$ be a path, $\sigma \equiv 0$ and $V \equiv -c$ where $c$ is a constant satisfying $0 < c < \lambda_{1,0} (= \lambda_1)$. From the definition of $\lambda_{1,V}$, it follows that
\[
\lambda_{1,V} \geq \lambda_{1,0} + \sum_{x \in S} V(x)|\phi_1|^p
\]
where $\phi_1$ is the positive eigenfunction corresponding to $\lambda_{1,V}$ satisfying $\int_S |\phi_1|^p = 1$. Hence $\lambda_{1,V} > \lambda_{1,0} > 0$. We now take a function $f$ satisfying
\[
\lambda_{1,0} \leq \limsup_{t \to \infty} \frac{f(x, t)}{t^{p-1}} < \lambda_{1,V}, \quad x \in S.
\]
Then it follows from Theorem 3.5 that there exists a positive solution to (1.6). On the other hand, it follows from the assumptions of $V$ and $f$ that
\[
\limsup_{t \to \infty} \frac{h(x, t)}{t^{p-1}} \geq \lambda_{1,0} + c, \quad x \in S
\]
where $h(x, t) := f(x, t) - V(x)t^{p-1}$. Hence in this example, $h$ does not satisfy the hypotheses in [1, Theorem 1.1] but (3.16) has a positive solution.
4. TWO POSITIVE SOLUTIONS

In this section, we prove that (1.6) has at least two positive solutions if \( (H1), \ (H2), \ (H3)', \) and \( (H4) \) hold. From now on, we assume that the function \( w_0 \) in \( (H3)' \) is not a solution to (1.6) and the definitions of \( \epsilon_0 \) and \( u_0 \) are the same as ones in Theorem 3.5.

**Theorem 4.6.** If we assume \( (H1), \ (H2), \) and \( (H3)' \) hold, then (1.6) has a positive solution \( u \) satisfying

\[
\epsilon_0^{-1} u_0(x) < u(x) < w_0(x), \quad x \in S. 
\]

**Proof.** For a function \( u \) defined in the proof of Theorem 3.5, we define a function \( \tilde{f} : S \times \mathbb{R} \to \mathbb{R} \) by

\[
\tilde{f}(x,t) := \begin{cases} f(x,u(x)), & t \leq u(x) \\ f(x,t), & u(x) < t < w_0(x) \\ f(x,w_0(x)), & w_0(x) \leq t. \end{cases}
\]

Then it is clear that

\[
\limsup_{|t| \to \infty} \frac{\tilde{f}(x,t)}{|t|^{p-2}} < \lambda_{1,V}.
\]

Hence by Lemma 3.3, there exists a global minimizer \( u_1 \) of \( E_{\tilde{f}} \). Since \( E_{\tilde{f}} \) is differentiable, the function \( u_1 \) is a solution to

\[
\begin{cases} -\Delta_p u(x) + V(x)|u(x)|^{p-2}u(x) = \tilde{f}(x,u(x)), & x \in S \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}
\]

By using the argument in Theorem 3.5, we can show \( u \not< u_1 \). Finally, we show that \( u_1(x) < w_0(x), \ x \in S \) by contradiction. Define a set \( \tilde{T} := \{ x \in S \mid u_1(x) \geq w_0(x) \} \) satisfying that an induced subgraph \( G(T, E') \) of \( \tilde{S}(V, E) \) is connected and that \( u_1(x) < w_0(x), \ x \in \partial T := \{ x \in \tilde{S} \setminus T \mid x \sim y \text{ for some } y \in T \} \). In the case of \( T = S \), by the definition of \( \tilde{f} \), we have

\[
-\Delta_p u_1(x) + V(x)|u_1(x)|^{p-2}u_1(x) = \tilde{f}(x,u_1(x)) = f(x,w_0(x)) 
\]

\[
\leq -\Delta_p w_0(x) + V(x)|w_0(x)|^{p-2}w_0(x)
\]

for all \( x \in S \) and \( u_1(x) = w_0(x) \) for all \( x \in \partial S \). Hence by the comparison principle, \( u_1 \equiv w_0 \) which is a contradiction. Moreover, in the case of \( T \neq S \), there exists \( x_0 \in S \cap \partial T \) such that \( u_1(x_0) < w_0(x_0) \). Hence by the comparison principle, \( u_1(x) < w_0(x), \ x \in T \), which is also a contradiction. Thus we get the desired result. \( \square \)

We now discuss the existence of the second solution to (1.6). To prove this, we first present a condition of \( g \in C(S \times \mathbb{R}; \mathbb{R}) \) which implies that the functional \( E_g \) satisfies the Palais-Smale condition (simply, (PS) condition):
(PS) Suppose that $\Omega$ is a real Banach space. A functional $E \in C^1(\Omega; \mathbb{R})$ satisfies the Palais-Smale condition if for any sequence $(u_n) \subset \Omega$ satisfying
(a) $E[u_n]$ is bounded and
(b) $E'[u_n] \to 0$ as $n \to \infty$,
the sequence $(u_n)$ has a convergent subsequence. A sequence satisfying (a) and (b) is called a (PS) sequence for $E$.

Lemma 4.7. Let (H1) hold and a function $g \in C(S \times \mathbb{R}; \mathbb{R})$ satisfies
\[
\limsup_{t \to -\infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S.
\]
for all $x \in S$. Then the functional $E_g$ satisfies the (PS) condition.

Proof. Let \( \{u_n \in A_\sigma\} \) be a (PS) sequence for $E_g$. Suppose that $\|u_n\|_p \to \infty$ where $u_n(x) := \max\{-u_n(x), 0\}$, $x \in S$. We note that $u_n(x) = 0$ for all $x \in \partial S$, namely, $u_n \in A_0$. Define a function $\epsilon_n : S \to \mathbb{R}$ by
\[
\epsilon_n(x) := -\Delta_{p,\omega} u_n(x) + V(x)|u_n(x)|^{p-2}u_n(x) - g(x, u_n(x)), \quad x \in S.
\]
Then we have $\epsilon_n \to 0$ by the definition of a (PS) sequence. Since $g$ satisfies
\[
\limsup_{t \to -\infty} \frac{g(x,t)}{|t|^{p-2}t} < \lambda_{1,V}, \quad x \in S,
\]
there exists a real value $M < \lambda_{1,V}$ such that
\[
g(x,t) > M|t|^{p-2}t + C, \quad t > 0
\]
for some constant $C$. It follows from the definition of $\epsilon_n$ and (4.18) that
\[
0 = \int_S \left[-\Delta_{p,\omega} u_n(x) + V(x)|u_n(x)|^{p-2}u_n(x) - g(x, u_n(x)) - \epsilon_n(x)\right] u_n^-(x)
\leq \lambda_{1,V} - M\|u_n\|_p - \int_S (C + \epsilon_n)u_n^-
\to -\infty, \quad n \to \infty,
\]
an obvious contradiction. Hence $\{u_n^-\}$ is bounded. We now suppose that $\|u_n\|_p \to \infty$. Define a function $v_n \in A_\sigma$ by
\[
v_n(x) := \frac{u_n(x)}{\|u_n\|_p}, \quad x \in S.
\]
Then there exists a function $v_0 \in A_\sigma$ such that $v_n \to v_0$. Moreover, since $\{u_n^-\}$ is bounded and $\|u_n\|_p \to \infty$, $v_0 \geq 0$ and $v_0 \neq 0$. It follows from the assumption
\[
\lambda_{1,V} < \liminf_{t \to \infty} \frac{g(x,t)}{|t|^{p-2}t}, \quad x \in S
\]
that for sufficiently small $\epsilon > 0$,

$$g(x,t) > (\lambda_{1,V} + \epsilon)\|t\|^{p-2}t + C, \quad t > 0, \quad x \in S$$

for some constant $C$. Hence we have that

$$-\Delta_{p,\omega}v_0(x) + V(x)|v_0(x)|^{p-2}v_0(x) \geq (\lambda_{1,V} + \epsilon)|v_0(x)|^{p-2}v_0(x)$$

for all $x \in S$. Therefore the function $v_0$ is a supersolution to

$$\tag{4.19} \begin{cases} -\Delta_{p,\omega}u(x) + V(x)|u(x)|^{p-2}u(x) = (\lambda_{1,V} + \epsilon)|u(x)|^{p-2}u(x), & x \in S, \\ u(x) = \sigma(x), & x \in \partial S. \end{cases}$$

Moreover, since the function $v_0$ satisfies

$$\begin{cases} -\Delta_{p,\omega}v_0(x) + V(x)|v_0(x)|^{p-2}v_0(x) \geq 0, & x \in S, \\ v_0(x) \geq 0, & x \in \partial S \end{cases}$$

and $v_0(x) > 0$ for some $x \in S$. It follows from the comparison principle that $v_0(x) > 0$ for all $x \in S$. Hence the positive eigenfunction $\phi_1$, corresponding to $\lambda_{1,V}$, is a subsolution to (4.19). Moreover, by the comparison principle, $v_0(x) > \phi_1(x)$ for all $x \in S$. Thus by the sub-super solution method, for $\epsilon > 0$, the equation (4.19) has a solution which implies $\lambda_{1,V}$ is not isolated, which contradicts. Hence $\{u_n\}$ is bounded. Thus $E_f$ satisfies the (PS) condition. \qed

We proved that there exists a positive solution to (1.6) in Theorem 4.6. The next result shows the existence of another positive solution.

**Theorem 4.8.** Suppose that the hypotheses in Theorem 4.6 hold. If, in addition, (H4) holds, then (1.6) has at least two positive solutions $u_1$ and $u_2$ satisfying $u_1 < u_2$.

**Proof.** It follows from Theorem 4.6 that there exists a positive solution $u_1 < u_0$. Now, we define a function $f_1 \in C(S \times \mathbb{R}; \mathbb{R})$ by

$$\tilde{f}_1(x,t) := \begin{cases} f(x,u_1(x)), & t \leq u_1(x) \\ f(x,t), & u_1(x) < t < u_0(x) \\ f(x,w_0(x)), & t \geq w_0(x) \end{cases}$$

for $x \in S$. Then it is clear that the function $f_1$ satisfies

$$\limsup_{\|t\| \to \infty} \frac{\tilde{f}_1(x,t)}{\|t\|^{p-2}t} < \lambda_{1,V}.$$ 

Thus by Lemma 3.3, the functional $E_{\tilde{f}_1}$ has a global minimizer $v_0$. Using the similar argument in the proof of Theorem 4.6, we have $u_1 < v_0 < w_0$ on $S$. We define a function $f_1 \in C(S \times \mathbb{R}; \mathbb{R})$ by

$$f_1(x,t) := \begin{cases} f(x,u_1(x)), & t \leq u_1(x) \\ f(x,t), & u_1(x) < t \end{cases}$$
Then the function $f_1$ satisfies
\[
\limsup_{t \to -\infty} \frac{f_1(x,t)}{|t|^{p-2}t} < \lambda_{1,V} < \liminf_{t \to \infty} \frac{f_1(x,t)}{|t|^{p-2}t}
\]
Thus $E_{f_1}$ satisfies the (PS) condition. Moreover, It follows from (H4) that there exists $M > \lambda_{1,V}$ such that
\[
E_{f_1}[t\phi_1] \leq \lambda_{1,V} - M t^p - Ct \to -\infty \text{ as } t \to \infty
\]
for some constant $C$. Since $E_{\tilde{f}_1}[u] = E_{f_1}[u]$ for all $u_1 \leq u < w_0$, the function $v_0$ is a local minimizer of $E_{f_1}$. Hence by Mountain Pass Theorem, there exists a critical point $u_2$ of $E_{f_1}$. Using the comparison principle, we have $u_1 < u_2$.

5. EXAMPLES

In this section, we give some corollaries and examples for results obtained in Section 4.

**Corollary 5.9.** Suppose that (H1) and (H2) hold and that there exists $t_1 > \max_{z \in \partial S} \sigma(z)$ such that $f(x,t_1) \leq 0$, $x \in S$. If the function $V$ satisfies
\[
V(x) \geq -\left(1 - \frac{\max_{y \in \partial S} \sigma(y) + \delta}{t_1}\right)^{p-1} \sum_{y \in \partial S} \omega(x,y), \quad x \in S
\]
where $\delta \in (0, t_1 - \max_{z \in \partial S} \sigma(z))$ then (1.6) has a solution $u$ such that
\[
\epsilon_0^{p-1} u_0(x) < u(x) < t_1, \quad x \in S.
\]

**Proof.** We put a function $\tau_1(x) = t_1$ for all $x \in S$ and $\tau_1(x) = \sigma(x) + \delta$ for all $x \in \partial S$. Then since $f(x,t_1) \leq 0$ for all $x \in S$, we have
\[
-\Delta_p \omega \tau_1 + V(x) |\tau_1|^{p-2} \tau_1 \geq (t_1 - (\max_{y \in \partial S} \sigma(y) + \delta))^{p-1} \sum_{y \in \partial S} \omega(x,y) + V(x) t_1^{p-1}
\]
for all $x \in S$. Since $\tau_1(x) > \sigma(x)$ for all $x \in \partial S$, the function $\tau_1$ is a supersolution to (1.6). Hence by Theorem 4.6, we have the desired result.

**Example 3.** We start this example with a set $\partial^o S$ defined by
\[
\partial^o S := \{x \in S \mid \omega(x,y) > 0 \text{ for some } y \in \partial S\}.
\]
Let a graph $G$ be a path, $\sigma \equiv 0$ and
\[
V(x) := \begin{cases} 0, & x \in S \setminus \partial^o S, \\ -\alpha, & x \in \partial^o S \end{cases}
\]
where $-\alpha$ is a negative value greater than the right hand side of (5.20) and $-\lambda_1$. Then it is easily proved that $\lambda_{1,V} > 0$. We now take a function $f$ satisfying
(i) \( f(x, t) = 0 \) for all \( t \) and \( x \in S \setminus \partial S \);
(ii) \( f(x, 0) > 0 \) for all \( x \in \partial S \);
(iii) \( f(x, t) > -\alpha t^{p-1} \) for all \( t > 0 \) and \( x \in \partial S \);
(iv) \( 0 > f(x, t_1) \) for all \( x \in \partial S \).

Then there exists a positive solution to (1.6) by Corollary 5.9.

Now taking \( h(x, t) := f(x, t) - V(x)|t|^{p-2} \), we have the equation

\[
\begin{align*}
-\Delta_{p, \omega} u(x) &= h(x, u(x)), \quad x \in S \\
u(x) &= 0, \quad x \in \partial S.
\end{align*}
\]

Then \( h(x, t) > 0, x \in \partial S, t > 0 \). Hence the function \( h \) does not satisfy the condition (1.4) in [1].

In addition, we show one more example in respect of a positive function \( f \). To construct the function \( f \), we introduce the eigenvalue problem without Dirichlet boundary condition:

\[-\Delta_{p, \omega} \psi(x) + \nabla(x)|\psi(x)|^{p-2} \psi(x) = \mu |\psi(x)|^{p-2} \psi(x), \quad x \in \overline{S}\]

for a function \( \nabla : \overline{S} \to \mathbb{R} \). We note that this eigenvalue problem has the first eigenvalue \( \mu_{1, \nabla} \) which is given by

\[
\mu_{1, \nabla} := \inf_{\phi \neq 0} \frac{\frac{1}{2} \int_{S} \nabla \phi \cdot \nabla_{p, \omega} \phi + \int_{S} |\nabla \phi|^p}{\int_{S} |\phi|^p}
\]

and there exists a positive eigenfunction \( \psi_1 \) corresponding to \( \mu_{1, \nabla} \). We note that if \( \nabla \equiv 0 \), then \( \mu_{1, \nabla} = 0 \) and \( \psi_1(x) = \psi_1(y), \, x, y \in \overline{S} \). Moreover, the multiplicity of \( \mu_{1, \nabla} \) is one and \( \mu_{1, \nabla} \) is isolated (for more details, see [10]).

We now show the next corollary with a definition of a function \( \nabla : \overline{S} \to \mathbb{R} \) as follows: for a given \( V : S \to \mathbb{R} \),

\[
\nabla(x) := \begin{cases} 
V(x), & x \in S \\
k, & x \in \partial S
\end{cases}
\]

where \( k \) is a constant.

**Corollary 5.10.** Let \((H1)\) and \((H2)\) hold. Suppose that \( \mu_{1, \nabla}, \psi_1, \) and \( f \) satisfy

\[
\psi_1(x) > \max_{x \in \partial S} \sigma(x), \quad x \in \overline{S}
\]

and

\[
f(x, \psi_1(x)) \leq \mu_{1, \nabla} \psi_1^{p-1}(x), \quad x \in S.
\]

Then (1.6) has a solution \( u \) such that

\[
\epsilon_0^{\frac{1}{p-1}} u_0(x) < u(x) < \psi_1(x), \quad x \in S.
\]
Proof. It follows from (5.23), (5.24), and the definition of $\mu_{1,\varphi}$ that the eigenfunction $\psi_1$ is a supersolution but a solution to (1.6). Hence by Theorem 4.6, this corollary is proved.

Example 4. Suppose that a graph $G$ is a path, $V$ is positive and non-constant and $\sigma \equiv 0$. Then by definitions of $\lambda_{1,V}$, we have $\lambda_{1,V} > 0$. In addition, we take $k > \max_{x \in S} V(x)$, then $\mu_{1,\varphi} > 0$. Since $V$ is non-constant, it follows from the definition of $\mu_{1,\varphi}$ that

$$\mu_{1,\varphi} > V(x_0) = \min_{x \in S} V(x).$$

We now take a function $f$ satisfying (5.24) and $f(x,t) > 0$ for $x \in S$ and $t > 0$. Then by Corollary 5.10, there exists a positive solution to (1.6). Finally, since $\psi_1$ also become a supersolution to (1.6), there exists the second positive solution $u_2$ by Theorem 4.8 if (H4) holds.

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