DISCRETE HERMITE-HADAMARD INEQUALITY
AND ITS APPLICATIONS

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We state and prove the Hermite-Hadamard inequality for a function defined on a time scale, which has all the points as isolated. We illustrate our result with various examples.

1. INTRODUCTION

The Hermite-Hadamard inequality [7, 8] states that if f : I → ℝ is a convex function, then the following inequality is satisfied:

\[ f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \left( \int_a^b f(t) \, dt \right) \leq \frac{f(a) + f(b)}{2}, \]

where a, b ∈ I and I is an interval in ℝ.

In the theory of convex functions, the Hermite-Hadamard inequality plays an important role. It has been used as a tool to obtain many nice results in integral inequalities, approximation theory, optimization theory and numerical analysis. It has been developed for different classes of convexity, such as quasi-convex functions, log-convex, r-convex functions, p-functions [6], and recently for discrete functions [1]. For the history of its developments in many directions, we refer the reader to a paper by Mitrinović and Lacković [10]. For the generalizations and applications in probability, we refer the reader to a paper by Merkle [9].

Mozyrska and Torres first introduced the convexity of a function defined on a time scale (a nonempty closed subset of ℝ) in their paper [11]. This short paper can be considered as the establishment of the foundation of convex functions on time scales.

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Here we introduce convexity by means of a midpoint condition of a function defined on a time scale which has all the points as isolated. We state and prove the Hermite-Hadamard inequality for such a class of functions. We call this new inequality as discrete Hermite-Hadamard inequality.

The paper is organized as follows: In Section 2, we define a convex real valued function on a discrete time scale $\mathbb{T}$ which time points may not be uniformly distributed on a time line. We state the midpoint condition for a function defined on $\mathbb{T}$. We then prove four equivalent statements for convex functions on $\mathbb{T}$. Section 3 starts with the proof of one kind of a substitution rule for integrals on any time scale. Then with the use of the substitution rules we prove the Hermite-Hadamard inequality for convex functions defined on $\mathbb{T}$. Three corollaries follow the main result of the paper. In the third corollary, we give an alternate proof to the Hermite-Hadamard inequality for functions defined on the set of real numbers $\mathbb{R}$. Last section of the paper is devoted to some interesting examples of the Hermite-Hadamard inequality. We close the paper with two remarks for the future direction of the research in this area.

2. PRELIMINARIES

Let $\mathbb{T} = \{0 = t_0, t_1, t_2, t_3, \ldots \}$ be the set of positive real numbers such that $t_i < t_j$ for $i < j$. We assume that $|\mathbb{T}| = \aleph_0$. We then define the operators $\sigma(t_i) = t_{i+1}$, $\mu(t_i) = \sigma(t_i) - t_i$, $\rho(t_i) = t_{i-1}$, and $\nu(t_i) = t_i - \rho(t_i)$ for $t_i \in \mathbb{T}$, which are known as the forward jump, the forward graininess, the backward jump, and the backward graininess operators, respectively. The time scale $\mathbb{T}$ can be considered as a discrete time scale. If $\mathbb{T}$ is the set of integers (i.e. $\mathbb{T} = \mathbb{Z}$), then $\sigma(t) = t + 1$, $\mu(t) = 1$, $\rho(t) = t - 1$ and $\nu(t) = 1$, for all $t \in \mathbb{T}$.

Definition 1. Let $f$ be a real valued function defined on $\mathbb{T}$. Then the $\Delta$-derivative and the $\nabla$-derivative of $f$ are defined, respectively, as

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}, \quad f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)},$$

where $t \in \mathbb{T}$. We define a second order derivative as $f^{\Delta \Delta} := (f^\Delta)^\Delta$. The $\Delta$-integral and the $\nabla$-integral of $f$ are defined as

$$\int_a^b f(\tau) \Delta \tau = \sum_{s \in [a,b) \cap \mathbb{T}} f(s) \mu(s), \quad \int_a^b f(\tau) \nabla \tau = \sum_{s \in (a,b] \cap \mathbb{T}} f(s) \nu(s),$$

respectively, where $a, b \in \mathbb{T}$.

We also use the notation $\mathbb{T}^\kappa$ which is defined as

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \sup \mathbb{T} < \infty \\ \mathbb{T}, & \sup \mathbb{T} = \infty. \end{cases}$$
For further reading on time scales, we refer the reader to an excellent book on the analysis of time scales [2].

Throughout this study, we focus on the discrete time scales. Let $T$ be any discrete time scale and $a, b \in T$ with $a < b$. $[a, b]_T$ means $[a, b] \cap T$. We define

$$T_{[a, b]} = \{ t \mid t = \frac{b - u}{b - a} \text{ for } u \in [a, b]_T \}.$$  

Note that $T_{[a, b]} \subset [0, 1]$. We also want to point out that there exists a bijective (one-to-one and onto) map between $[a, b]_T$ and $T_{[a, b]}$.

**Definition 2.** $f : T \rightarrow \mathbb{R}$ is called convex on $T$ if for every $x, y \in T$ with $x < y$, the following inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$  

is satisfied for all $\lambda \in T_{[x, y]}$.

**Definition 3.** We define the midpoint of $a$ and $b$ as the $n^{th}$ element in a finite time scale $[a, b]_T$ with cardinality $2n - 1$.

For the notation, we denote the midpoint of $a$ and $b$ by $m_{[a, b]}$ in $[a, b]_T$, and the corresponding midpoint of 0 and 1 by $m_{[0, 1]}$ in $T_{[a, b]}$. If $T = \mathbb{Z}$, then Definition 3 reduces to the standard definition of midpoint on $\mathbb{Z}$, namely $m_{[a, b]} = \frac{a + b}{2}$ and $m_{[0, 1]} = \frac{1}{2}$.

**Definition 4.** $f : T \rightarrow \mathbb{R}$ satisfies the midpoint condition if

$$(1) \quad f(m_{[a, b]}) \leq m_{[0, 1]}f(a) + (1 - m_{[0, 1]})f(b),$$  

for every $a, b \in T$ with the cardinality of $[a, b]_T$ an odd number.

**Remark 1.** If we want to be more precise for the number $m_{[0, 1]}$ in (1), then we can write it as $m_{[0, 1]} = \frac{b - m_{[a, b]}}{b - a}$. Note that if $T = \mathbb{Z}$, then the inequality (1) becomes

$$f\left( \frac{a + b}{2} \right) \leq \frac{f(a) + f(b)}{2},$$  

as stated in [1].

The following two theorems are crucial in the proof of the next theorem about some equivalent criteria of convexity for real valued functions defined on $T$.

**Theorem 5 (Taylor’s Theorem [2]).** Let $n \in \mathbb{N}$. Suppose that $f$ is $n$-times differentiable on $\mathbb{T}^{n-1}$. Let $\alpha \in \mathbb{T}^{n-1}$, $t \in T$, and define the functions $h_k$ by

$$h_0(r, s) \equiv 1 \text{ and } h_{k+1}(r, s) = \int_s^r h_k(\tau, s) \Delta \tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t, \alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\varphi^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta \tau.$$
**Theorem 6** (Mean Value Theorem [3]). Let \( f \) be a continuous function on \([a, b]_T\) that is differentiable on \([a, b]_T\). Then there exists \( \xi, \tau \in [a, b]_T \) such that
\[
f^\Delta(\tau) \leq \frac{f(b) - f(a)}{b - a} \leq f^\Delta(\xi).
\]

**Theorem 7.** Let \( f : \mathbb{T} \to \mathbb{R} \). The following are equivalent:

(i) \( f \) is convex on \( \mathbb{T} \).

(ii) \( f \) satisfies the midpoint condition (1).

(iii) \( f^{\Delta^2}(t) \geq 0 \) for all \( t \in \mathbb{T} \).

(iv) \( f(x) \geq f(y) + (x - y)f^{\Delta}(y) \) for all \( x, y \in \mathbb{T} \) with \( x > y \),

\( \text{or} f(x) \geq f(y) + (x - y)f^{\nabla}(y) \) for all \( x < y \).

**Proof.** We prove that (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i) \( \Rightarrow \) (iv) \( \Rightarrow \) (iii).

(i) \( \Rightarrow \) (ii): Let \( a, b \in \mathbb{T} \) with \( a < b \) and \([a, b]_T\) have odd number of time points. This implies that \([a, b]_T\) has a midpoint \( m_{[a,b]} \). Since \( f \) is convex, by choosing \( \lambda = \frac{b - m_{[a,b]}}{b - a} \), we obtain
\[
f(m_{[a,b]}) \leq \left( \frac{b - m_{[a,b]}}{b - a} \right) f(a) + \left( \frac{m_{[a,b]} - a}{b - a} \right) f(b).
\]

Then the midpoint condition (1) follows.

Next we prove that (ii) implies (iii). Let \( t \in \mathbb{T} \). Then \( \sigma(t) \in \mathbb{T} \). Applying the midpoint condition at \( \sigma(t) \), we have
\[
f(\sigma(t)) \leq \frac{\sigma^2(t) - \sigma(t)}{\sigma^2(t) - t} f(t) + \frac{\sigma(t) - t}{\sigma^2(t) - t} f(\sigma^2(t)).
\]

Simple algebra implies that \( f^{\Delta^2}(t) \geq 0 \).

Next we prove that (iii) \( \Rightarrow \) (i). Let \( x, y \in \mathbb{T} \) with \( x < y \). Fix \( \lambda \in \mathbb{T}_{[x,y]} \).

Define \( x_0 = \lambda x + (1 - \lambda)y \). Using Taylor’s theorem (Theorem 5) at \( x_0 \) we have
\[
f(y) = \sum_{i=0}^{1} h_i(y, x_0)f^{\Delta^i}(x_0) + \sum_{\tau = x_0}^{\rho^2(y)} h_1(y, \sigma(\tau))f^{\Delta^2}(\tau)(\sigma(\tau) - \tau).
\]

Since \( f^{\Delta^2}(\tau) \geq 0 \) on \( \mathbb{T} \) and \( h_1(y, \sigma(\tau)) = y - \sigma(\tau) \geq 0 \) on \( \mathbb{T} \), we have
\[
(2) \quad f(y) \geq f(x_0) + (y - x_0)f^{\Delta}(x_0).
\]

Using the Mean Value Theorem (Theorem 6) for \( f \) on \([x, x_0]_T\), there exists \( \tau \in [x, x_0]_T \) such that
\[
\frac{f(x_0) - f(x)}{x_0 - x} \leq f^{\Delta}(\tau).
\]
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Since \( f^\Delta^2(t) \geq 0 \) on \( \mathbb{T} \), we have \( f^\Delta(\tau) \leq f^\Delta(x_0) \). Therefore we obtain

\[
(3) \quad f(x) \geq f(x_0) + (x - x_0) f^\Delta(x_0).
\]

If we multiply the inequality (2) by \( 1 - \lambda \) and the inequality (3) by \( \lambda \) and add them side by side, we obtain

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]

for all \( \lambda \in \mathbb{T}_{[x,y]} \).

Now we prove (i) \( \Rightarrow \) (iv). Given any \( x, y \in \mathbb{T} \) such that \( y < x \), by convexity of \( f \) on \( [y,x] \), we have

\[
f(\sigma(y)) \leq \frac{\sigma(y) - y}{x - y} f(x) + \frac{x - \sigma(y)}{x - y} f(y).
\]

After rearranging the terms we have

\[
f^\Delta(y) \leq \frac{f(x) - f(y)}{x - y}.
\]

This simplifies to \( f(x) \geq f(y) + (x - y)f^\Delta(y) \) for all \( x > y \). The same argument works to show that

\[
f(x) \geq f(y) + (x - y)f^\nabla(y) \quad \text{for all} \quad x < y.
\]

Finally we prove (iv) \( \Rightarrow \) (iii). Given \( f(x) \geq f(y) + (x - y)f^\Delta(y) \) for all \( x > y \). By choosing \( x = \sigma^2(y) \), we obtain \( f^\Delta^2(y) \geq 0 \). Since \( y \) is arbitrary, (iii) follows.

This completes the proof.

3. HERMITE-HADAMARD INEQUALITY ON DISCRETE TIME SCALES

In this section we prove discrete Hermite-Hadamard inequality for convex functions defined on a discrete time scale \( \mathbb{T} \).

**Theorem 8** (Substitution rule on time scales [5]). Assume \( \nu : \mathbb{T} \to \mathbb{R} \) is strictly increasing and \( \mathbb{T} := \nu(\mathbb{T}) \) is a time scale. If \( f : \mathbb{T} \to \mathbb{R} \) is a \( rd \)-continuous function and \( \nu \) is differentiable with \( rd \)-continuous derivative, then if \( a, b \in \mathbb{T} \),

\[
\int_a^b f(t)^\Delta(t) \Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s) \Delta s
\]

or

\[
\int_a^b f(t)^\nabla(t) \nabla t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s) \nabla s.
\]

Next we state and prove the substitution rule for a strictly decreasing function \( \nu : \mathbb{T} \to \mathbb{R} \), where \( \mathbb{T} \) can now be any time scale, and not strictly isolated.
Theorem 9. Assume $\nu : \mathbb{T} \to \mathbb{R}$ be strictly decreasing and $\tilde{T} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is a continuous function and $\nu$ is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,

$$\int_a^b f(t)(-\nu^\Delta)(t) \Delta t = \int_{\nu(b)}^{\nu(a)} (f \circ \nu^{-1})(s) \nabla s$$

and

$$\int_a^b f(t)(-\nu^\nabla)(t) \nabla t = \int_{\nu(b)}^{\nu(a)} (f \circ \nu^{-1})(s) \tilde{\Delta} s.$$ 

Proof. We prove the second equality. In the proof we use the basics of the dual time scales introduced in [4] and the substitution method for a strictly increasing function (Theorem 8). We start on the right side of the equality:

$$\int_{\nu(b)}^{\nu(a)} (f \circ \nu^{-1})(s) \tilde{\Delta} s = \int_{\nu^{-1}(b)}^{\nu^{-1}(a)} f(u^{-1} \circ \nu)(s) \nabla s$$

$$= \int_{\nu^{-1}(b)}^{\nu^{-1}(a)} f(u^{-1} \circ \nu)(s) \nabla s$$

$$= \int_a^b f(t)(u^{-1} \circ \nu)^\nabla(t) \nabla t = \int_a^b f(t)(-\nu)^\nabla t,$$

where $\mathbb{T}$ represents the dual time scale, $u(s) := -s$ and $f((\nu^{-1})^\nabla(s) = f((\nu^{-1} \circ u)(s)) = f((u^{-1} \circ \nu)^{-1}(s)).$

Remark 2. We note that the statement of Theorem 2.3 (ii) in the paper [5] is not correct since $\tilde{T}$ was defined as $-\nu(\mathbb{T})$ for a strictly decreasing function $\nu.$

We use the following notation in the proof of the next theorem: Let $a, b \in \mathbb{T}, a < b$ and the cardinality of $[a, b]_\mathbb{T}$ be an odd number, say $k + 1. \ Let t \in T_{[a, b]}$. Then there exists an $n \in \mathbb{N} \cup \{0\}$ such that $t = \sigma^n(0).$ We denote $\hat{t}$ by $\sigma^{k-n}(0).$ Similarly, let $u \in [a, b]_\mathbb{T}.$ Then there exists a $\ell \in \mathbb{N} \cup \{0\}$ such that $u = \sigma^{\ell}(a).$ We denote $\hat{u}$ by $\sigma^{k-\ell}(a).$ We also note that $\hat{u} = u$ and $\hat{t} = t.$

Next we illustrate this new notation with an example.

Example 1. Let $[a, b]_\mathbb{T} = \{x_0 = a, x_1, x_2, x_3, x_4, x_5, x_6 = b\}$, where $x_i < x_{i+1}$ for $0 \leq i \leq 5.$ Then we have $\tilde{x}_i = \sigma^{k-i}(a) = x_{a-i}$ for $0 \leq i \leq 6.$ It follows that

$$T_{[a, b]} = \{t_0 = 0, t_1 = \frac{b - x_5}{b - a}, t_2 = \frac{b - x_4}{b - a}, t_3 = \frac{b - x_3}{b - a}, t_4 = \frac{b - x_2}{b - a}, t_5 = \frac{b - x_1}{b - a}, t_6 = 1\}.$$ 

Hence we have $\tilde{t}_i = \sigma^{6-i}(0) = t_{6-i}$ for $0 \leq i \leq 6.$

This implies that $\tilde{t}_i = \frac{b - x_i}{b - a}.$ One simple algebra step implies that

$$x_i = a\tilde{t}_i + (1 - \tilde{t}_i)b.$$ 

On the other hand, $\tilde{x}_i = x_{6-i} = a\tilde{t}_i + (1 - \tilde{t}_i)b.$
Theorem 10. Suppose that \( f : \mathbb{T} \to \mathbb{R} \) is a convex function on \([a, b]_\mathbb{T}\). Then
\[
\hat{f}(m_{[a, b]}) \leq \frac{1}{b-a} \int_{[a, b]_\mathbb{T}} k(t) f(t) \nabla(t) - \frac{1}{b-a} \int_{[a, b]_\mathbb{T}} g^\Delta(t) k(t) f(t) \Delta t
\]
where \( g : [a, b]_\mathbb{T} \to [a, b]_\mathbb{T} \) is defined by \( g(u) = \hat{u} \) and \( k : [a, b]_\mathbb{T} \to \mathbb{R}^+ \) is defined by
\[
k(x) := \begin{cases} 
\frac{g(x) - m_{[a, b]}}{y(x) - x}, & x \neq m_{[a, b]} \\
1/2, & x = m_{[a, b]}.
\end{cases}
\]

Proof. Fix \( t \in T_{[a, b]} \). Then there exists \( x \in [a, b]_\mathbb{T} \) such that \( x = ta + (1 - t)b \). As we pointed out in Example 1, \( x = \hat{t}a + (1 - \hat{t})b \). Denote \( x \) by \( y \), i.e. \( y = \hat{t}a + (1 - \hat{t})b \). Note that \( m_{[a, b]} = m_{[x, y]} \) using the definition of the hat operator.

Let \( \xi : [a, b]_\mathbb{T} \to T_{[a, b]} \) be a linear function defined as \( \xi(u) = \frac{b - u}{b - a} \). Hence we have \( \xi(x) = t \) and \( \xi(y) = \hat{t} \). If \( x \neq m_{[a, b]} \), then by convexity of \( f \) we have
\[
f(m_{[a, b]}) \leq \frac{y - m_{[a, b]}}{y - x} f(x) + \frac{m_{[a, b]} - x}{y - x} f(y).
\]
If \( x = m_{[a, b]} \), then it reduces to \( x = y = m_{[a, b]} \). Clearly we have
\[
f(m_{[a, b]}) = \frac{1}{2} f(x) + \frac{1}{2} f(y).
\]

We combine (5) and (6) using the function \( k \)
\[
f(m_{[a, b]}) \leq k(x) f(x) + k(y) f(y).
\]

Next we integrate each side of the above inequality from 0 to 1 on \( T_{[a, b]} \) and we obtain
\[
\int_{T_{[a, b]}} \hat{f} \leq \int_{T_{[a, b]}} k(x) f(x) \hat{t} + \int_{T_{[a, b]}} k(y) f(y) \hat{t}
\]
\[
= \int_{T_{[a, b]}} k(\xi^{-1}(t)) f(\xi^{-1}(t)) \hat{t} + \int_{T_{[a, b]}} k(\xi^{-1}(\hat{t})) f(\xi^{-1}(\hat{t})) \hat{t}.
\]

Here we first claim that
\[
\int_{T_{[a, b]}} k(\xi^{-1}(t)) f(\xi^{-1}(t)) \hat{t} = \frac{1}{b-a} \int_{[a, b]_\mathbb{T}} k(t) f(t) \nabla t.
\]
Let’s define \( F := k \cdot f \). Then we have \( F(\xi^{-1}(t)) = k(\xi^{-1}(t)) f(\xi^{-1}(t)) \).

Next, we apply the substitution rule (Theorem 9) to the integral on the left side of the equality in (7).
\[
\int_{T_{[a, b]}} k(\xi^{-1}(t)) f(\xi^{-1}(t)) \hat{t} = \int_{T_{[a, b]}} F(\xi^{-1}(t)) \hat{t} = \int_{a}^{b} F(t) \frac{1}{b-a} \nabla t.
\]
This finishes the proof of our first claim.

Next we claim that

$$\int_{\mathcal{T}_{[a,b]}} k(\xi^{-1}(t)) f(\xi^{-1}(t)) \Delta t = -\frac{1}{b-a} \int_{[a,b]} g^\Delta(t) k(t) f(t) \Delta t. \quad (8)$$

Before we prove the equality (8), we point out that the function $g$ is a bijection and $g \equiv g^{-1}$ since $g^2$ is an identity function. As a result of this, we have $g(ta + (1-t)b) = ta + (1-t)b$. This observation will help us to complete the proof of the claim.

By applying $w(u) = \hat{\xi}(u)$ to the integral on the left side of the equality (8), we have

$$\int_{\mathcal{T}_{[a,b]}} k(\xi^{-1}(t)) f(\xi^{-1}(t)) \Delta t = \int_{\mathcal{T}_{[a,b]}} F(\xi^{-1}(t)) \Delta t$$

$$= \int_0^1 (F \circ w^{-1})(t) \Delta t = -\frac{1}{b-a} \int_{[a,b]} g^\Delta(t) F(t) \Delta t,$$

where

$$w^\Delta(u) = \frac{w(\sigma(u)) - w(u)}{\sigma(u) - u} = \frac{\xi(\sigma(u)) - \xi(u)}{\sigma(u) - u}$$

$$= \frac{b - \sigma(u)}{b - a} = \frac{b - \hat{u}}{b - a} = \frac{1}{a-b} \frac{\sigma(u) - \hat{u}}{\sigma(u) - u} = \frac{g^\Delta(u)}{a-b} \geq 0,$$

since $\sigma(u) < \hat{u}$. This completes the proof of our second claim.

To prove the right side of the inequality, we construct the following inequalities using convexity of $f$.

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b),$$

$$f(y) \leq \frac{b-y}{b-a} f(a) + \frac{y-a}{b-a} f(b).$$

Next, we multiply both inequality by $k(x)$ and $k(y)$ respectively. We obtain

$$k(x)f(x) \leq \frac{b-x}{b-a} k(x)f(a) + \frac{x-a}{b-a} k(x)f(b),$$

$$k(y)f(y) \leq \frac{b-y}{b-a} k(y)f(a) + \frac{y-a}{b-a} k(y)f(b).$$

Simple algebra implies the following identities

$$\frac{b-x}{b-a} k(x) + \frac{b-y}{b-a} k(y) = \frac{b - m_{[a,b]}}{b-a},$$

$$\frac{x-a}{b-a} k(x) + \frac{y-a}{b-a} k(y) = \frac{m_{[a,b]} - a}{b-a}.$$
Recall that \( x \) and \( y \) both depend on \( t \). We let \( t \) vary over \( \mathbb{T}_{[a,b]} \) and integrate each side of the last two inequalities on \( \mathbb{T}_{[a,b]} \) and we add them side by side, we obtain

\[
\int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(t))\frac{f(\nu^{-1}(t))\Delta t}{\nu^{-1}(t)} + \int_{\mathbb{T}_{[a,b]}} k(\nu^{-1}(t))\frac{f(\nu^{-1}(t))\Delta t}{\nu^{-1}(t)} \\
\leq \frac{b-a}{b-a} f(a) + \frac{a-b}{b-a} f(b) = m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b),
\]

where the last equality holds by means of Remark 1.

**Corollary 1.** Suppose that \( f : h\mathbb{Z} \to \mathbb{R} \) is a convex function with \( h > 0, a, b \in h\mathbb{Z}, a < b \). Then

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{2(b-a)} \left[ \int_{[a,b]_{h\mathbb{Z}}} f(t) \Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t) \nabla t \right] \leq \frac{f(a) + f(b)}{2}.
\]

**Proof.** Here \( g \) and \( k \) simplify into \( g(x) = a + b - x \) and \( k(x) = 1/2 \) and \( g^\Delta(x) = -1 \). Hence we have the desired inequality. \( \square \)

When \( h = 1 \), we obtain the Hermite-Hadamard inequality on \( \mathbb{Z} \).

**Corollary 2.** [1] Suppose that \( f : \mathbb{Z} \to \mathbb{R} \) is a convex function on \( [a,b]_{\mathbb{Z}} \) with \( a, b \in \mathbb{Z}, a < b, \) and \( a + b \) an even number. Then

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{2(b-a)} \left[ \int_{a}^{b} f(t) \Delta t + \int_{a}^{b} f(t) \nabla t \right] \leq \frac{f(a) + f(b)}{2}.
\]

Next we give an alternate proof of the Hermite-Hadamard inequality on \( \mathbb{R} \) (continuous Hermite-Hadamard inequality) by using the main result of this paper. For this purpose, we first state the following lemma without giving its proof.

**Lemma 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a convex function on \( \mathbb{R} \). Then for any \( h > 0 \), its restriction to \( h\mathbb{Z} \) is also a convex function.

**Corollary 3.** Let \( f \) be a real convex function on the finite interval \([a, b] \subset \mathbb{R}\). Then \( f \) satisfies the continuous Hermite-Hadamard inequality

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{2(b-a)} \left( \int_{a}^{b} f(t) dt \right) \leq \frac{f(a) + f(b)}{2}.
\]

**Proof.** By restricting \( f \) to \( h\mathbb{Z} \) we obtain the inequality (9). Since \( f \) is convex on \([a, b]\), it is continuous on \([a, b]\), hence integrable on \([a, b]\). When \( h \) tends to zero, the \( \Delta \)-integral and \( \nabla \)-integral converge to the Riemann integral of \( f \) on \([a, b]\). In other words,

\[
\lim_{h \to 0} \sum_{t \in [a,b]_{h\mathbb{Z}}} f(t)h = \int_{a}^{b} f(t)dt \text{ and } \lim_{h \to 0} \sum_{t \in [a,b]_{h\mathbb{Z}}} f(t)h = \int_{a}^{b} f(t)dt.
\]

Hence the result follows.
4. APPLICATIONS

(i). Let \( f(t) = (1 + h)^{t/h} \) be a function on \( h\mathbb{Z} \) for some positive real number \( h \).
Since \( f^{\Delta^2}(t) = f(t) \geq 0 \), \( f \) satisfies Hermite-Hadamard inequality on the interval \([a,b]_{h\mathbb{Z}}\), where \( a, b \in h\mathbb{Z} \).

\[
\int_{[a,b]_{h\mathbb{Z}}} f(t) \Delta t + \int_{[a,b]_{h\mathbb{Z}}} f(t) \nabla t \leq \frac{f(a) + f(b)}{2},
\]

where
\[
\int_{[a,b]_{h\mathbb{Z}}} f(t) \Delta t = \sum_{t \in [a,b]_{h\mathbb{Z}}} (1 + h)^{t/h}h = (1 + h)^b/h - (1 + h)^a/h,
\]
\[
\int_{[a,b]_{h\mathbb{Z}}} f(t) \nabla t = \sum_{t \in (a,b)_{h\mathbb{Z}}} (1 + h)^{t/h}h = (1 + h)^{1+b/h} - (1 + h)^{1+a/h}.
\]

Hence it follows that
\[
(1 + h) \frac{a+b}{2h} \leq \frac{1}{2(b-a)} \left( [(1 + h)^b/h - (1 + h)^a/h] + [(1 + h)^{1+b/h} - (a + h)^{1+a/h}] \right)
\]
\[
\leq \frac{(1 + h)^{a/h} + (1 + h)^{b/h}}{2}.
\]

Now, let \( x = f(a) \) and \( y = f(b) \), then the above inequality (11) simplifies into

\[
\sqrt{xy} \leq \frac{2 + h}{h} \left[ \frac{y - x}{f^{-1}(y) - f^{-1}(x)} \right] \leq \frac{x + y}{2}.
\]

for all \( x, y \in f(h\mathbb{Z}) \).
Note that inequality (12) holds for all \( x, y \in f(h\mathbb{Z}) \) for a given \( h > 0 \). Now we let \( h \) vary and take the limit as \( h \) goes to zero. Then \( f(t) = (1 + h)^{t/h} \) converges to \( e^t \) and the inequality turns into the well-known geometric-logarithmic-arithmetic mean inequality:
\[
\sqrt{xy} \leq \frac{y - x}{\ln y - \ln x} \leq \frac{x + y}{2},
\]
for all \( x, y \) positive real numbers.

(ii). Let \( f : h\mathbb{N} \to \mathbb{R} \) be defined as \( f(t) = \frac{t}{t} \) for some positive real number \( h \).
Since \( f^{\Delta^2}(t) = \frac{2}{(t+h)(t+2h)} \geq 0 \), \( f \) is convex on \( h\mathbb{N} \). Hence the Hermite-Hadamard inequality holds and we obtain
\[
\frac{2}{a+b} \leq \frac{H(b,a)}{b-a} - \frac{h}{2ab} \leq \frac{a + 1}{b + 1},
\]
where \( a, b \in h\mathbb{N} \) and \( H(b,a) = \int_{[a,b]_{h\mathbb{Z}}} \frac{1}{t} \Delta t \).
We take the limit as \( h \) goes to 0. Then we have
\[
\frac{2}{a + b} \leq \frac{\ln(b) - \ln(a)}{b - a} \leq \frac{1}{a} + \frac{1}{b},
\]
for all \( a, b \) positive real numbers.

**Theorem 11.** Let \( \mathbb{T} \) be a discrete time scale and \( f \) be function on \( \mathbb{T} \), not necessarily convex, satisfying \( \alpha \leq f^\Delta (t) \leq \beta \). Then we get
\[
\alpha U \leq \left( \frac{1}{b - a} \int_{[a,b]_\mathbb{T}} k(t) f(t) \nabla t - \frac{1}{b - a} \int_{[a,b]_\mathbb{T}} g^\Delta(t) k(t) f(t) \Delta t \right) - f(m_{[a,b]}) \leq \beta U
\]
\[
\alpha V \leq m_{[0,1]} f(a) + (1 - m_{[0,1]}) f(b) \cdot \left( \frac{1}{b - a} \int_{[a,b]_\mathbb{T}} k(t) f(t) \nabla t - \frac{1}{b - a} \int_{[a,b]_\mathbb{T}} g^\Delta(t) k(t) f(t) \Delta t \right) \leq \beta V,
\]
where
\[
U = \frac{1}{b - a} \left[ \int_{[a,b]_\mathbb{T}} k(t) h_2(t) \nabla t - \int_{[a,b]_\mathbb{T}} k(t) g^\Delta(t) h_2(t) \Delta t \right] - h_2(m_{[a,b]}) \quad \text{and}
\]
\[
V = m_{[0,1]} h_2(a) + (1 - m_{[0,1]}) h_2(b) \cdot \left[ \int_{[a,b]_\mathbb{T}} k(t) h_2(t) \nabla t - \int_{[a,b]_\mathbb{T}} k(t) g^\Delta(t) h_2(t) \Delta t \right].
\]

**Proof.** Let \( h_2(t) \) be the Taylor monomial with \( s = 0 \). In other words it is a function on \( \mathbb{T} \) whose second \( \Delta \)-derivative is 1. Taylor monomials are defined in Theorem 5. Let \( F(t) := f(t) - \alpha h_2(t) \) and \( G(t) := \beta h_2(t) - f(t) \). Since \( \alpha \leq F^\Delta(t) \leq \beta \) we have \( F^\Delta(t) \geq 0 \) and \( G^\Delta(t) \geq 0 \). Therefore \( F \) and \( G \) are convex. If we apply the Hermite Hadamard inequality for both \( F \) and \( G \), then we derive the desired inequalities.

**Corollary 4.** If \( \mathbb{T} = \mathbb{Z} \), then \( U = \frac{(b - a)^2 + 2}{24} \) and \( V = \frac{(b - a)^2 - 1}{12} \).

**Corollary 5.** If \( \mathbb{T} = \mathbb{Z}^q \), then
\[
U = \frac{1}{q(1 + q)^2(b - a)} \left[ q^2 \sqrt{ab}(b^2 - a^2) + 2n(q^2 - 1)(ab)^{3/2} - 2(q^2 + q)ab(b - a) \right] \quad \text{and}
\]
\[
V = \frac{1}{q(1 + q)^2(b - a)} \left[ q \sqrt{ab}(b^2 - a^2) - 2n(q^2 - 1)(ab)^{3/2} \right].
\]

The following two remarks will state some open problems for the researchers who are interested in working in this area.

**Remark 3.** Midpoint condition plays an important role to prove the main result of this paper. Even though the convexity of the function on any time scale has been defined in [1, 11], it is still an open problem to define the midpoint condition for such a function.
Remark 4. As it is pointed out in the paper [12], the Hermite-Hadamard inequality for the function defined on a real interval can characterize the convexity of the function. Again this can be done as another direction of the research for the class of functions we consider in this paper.

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REFERENCES


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