GENERAL AND ACYCLIC SUM-LIST-COLOURING OF GRAPHS

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We investigate list colouring of a graph in which the sizes of the lists assigned to different vertices can be different. For a given graph $G$ and a class of graphs $\mathcal{P}$ we colour $G$ from the lists in such a way that each colour class induces a graph in $\mathcal{P}$. The aim is to find the $\mathcal{P}$-sum-choice-number of $G$, which means the smallest possible sum of all the list sizes such that, according to the rules, $G$ is colourable for any particular assignment of the lists of these sizes. We prove several general results concerning the $\mathcal{P}$-sum-choice-number of an arbitrary graph. Using some of them, we also estimate or, in the case of complete graphs or some complete bipartite graphs, exactly determine the $\mathcal{P}$-sum-choice-number of a graph, when $\mathcal{P}$ is the class of acyclic graphs.

1. MOTIVATION AND PRELIMINARIES

The realities of the world around us provide many examples of tasks associated with a certain set of objects that are in some specific conflict relationships. In mathematics, such a structure is modelled by a graph. Obviously, the graph can be a model of a computer network or water supply as well as telecommunication, distribution and social networks. Network objects benefit from the access to the resources. In some simplifications, we can imagine that maintaining the availability of the resource is burdened by a unit cost. The aim of the study is to determine the smallest possible total cost of the availability of the resources throughout all the objects so that in any unit of time, by any allocation of the resources in accordance with the size of the access, the network works without any conflict. The mathematical description of this problem first appeared in 2002 [11] in connection to the studies on sum-list-colouring of graphs. This concept has generalized two well

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known concepts of list and sum colourings of graphs [6, 13]. An overview of the recent state of the research in this area is given in the Ph.D. Thesis of lastrina [14].

In the standard investigation of this type we assume that the set of objects is in conflict when it induces a graph with at least one edge. This paper deals with the general form of the conflict represented by an arbitrary but fixed graph or a family of graphs. In colouring terms we allow that vertices of one colour induce a graph that has some previously described properties, so it is not necessarily edgeless. Our investigation relates to the notions considered in the literature [2, 7, 9, 11, 12, 14].

Throughout this paper we follow the notation and terminology of [3, 4, 5]. In particular, we consider finite and undirected graphs $G$ with vertex set $V(G)$ and edge set $E(G)$ that are loopless and have no multiple edges. Let $G_1, G_2$ be graphs. The equality $G_1 = G_2$ means that $G_1, G_2$ are isomorphic, $G_2 \subseteq G_1$ denotes that $G_1$ contains an induced subgraph isomorphic to $G_2$, additionally, when this subgraph is proper we can write $G_2 < G_1$. The union $G_1 \cup G_2$ of two disjoint graphs $G_1, G_2$ is defined as a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For $p, q \in \mathbb{N}$ we denote by $K_p$ a $p$-vertex complete graph and by $K_{p,q}$ a $(p + q)$-vertex complete bipartite graph with partite sets of cardinality $p$ and $q$, respectively. The degree of a vertex $v$ in a graph $G$, $\text{deg}_G(v)$, is the number of edges incident with $v$ in $G$. By $\delta(G)$ we denote $\min\{\text{deg}_G(v) : v \in V(G)\}$.

A class of graphs $\mathcal{P}$ is induced hereditary if for each graph $G$ in $\mathcal{P}$ all its induced subgraphs are also in $\mathcal{P}$. An induced hereditary class of graphs $\mathcal{P}$ is additive when for any two graphs $G_1, G_2$ in $\mathcal{P}$ their union belongs to $\mathcal{P}$ too. Each induced hereditary class of graphs $\mathcal{P}$ can be uniquely characterized by family

$$\mathcal{F}(\mathcal{P}) = \{G : G \notin \mathcal{P} \text{ and } H \in \mathcal{P} \text{ for each } H < G\}.$$ 

By $\delta(\mathcal{P})$ we mean $\min\{\delta(G) : G \in \mathcal{F}(\mathcal{P})\}$. Note that for each induced hereditary class of graphs $\mathcal{P}$ it holds $K_1 \in \mathcal{P}$. To complete, we assume that an empty graph (without vertices and without edges) is also an element of each such class $\mathcal{P}$.

A list assignment $L$ of a graph $G$ is a collection $\{L(v)\}_{v \in V(G)}$ of nonempty subsets of set $\mathbb{N}$ of positive integers. Let $\mathcal{P}$ be an induced hereditary class of graphs. The graph $G$ is $(L, \mathcal{P})$-choosable if there exists a mapping (colouring) $c : V(G) \rightarrow \mathbb{N}$, such that $c(v) \in L(v)$ for each $v \in V(G)$ and for each $i \in \mathbb{N}$ the graph induced in $G$ by vertices coloured $i$ belongs to $\mathcal{P}$. Such a mapping $c$ is called $(L, \mathcal{P})$-colouring of $G$ or $\mathcal{P}$ sum-list-colouring of $G$ when we are not interested in the form of $L$. Next, let $f : V(G) \rightarrow \mathbb{N}$ be a function which assigns list sizes to the vertices of $G$ (in many cases $f$ will be called a size function (for $G$)). A graph $G$ is $(f, \mathcal{P})$-choosable if for every list assignment $L$ whose sizes are specified by $f$ ($|L(v)| = f(v)$ for all $v \in V(G)$) the graph $G$ is $(L, \mathcal{P})$-choosable.

The $\mathcal{P}$-sum-choice-number $\chi^\mathcal{P}_{\text{sc}}(G)$ of a graph $G$ is the minimum of the sum of sizes in $f$ over all $f$ such that $G$ is $(f, \mathcal{P})$-choosable. Thus

$$\chi^\mathcal{P}_{\text{sc}}(G) = \min \left\{ \sum_{v \in V(G)} f(v) : G \text{ is } (f, \mathcal{P}) - \text{choosable} \right\}.$$
Observe that if $P$ is the class $O$ of all edgeless graphs, then $\chi_\text{sc}^O(P) = \chi_\text{sc}(G) = \chi_\text{ac}(G)$, where the last symbol was introduced in [12]. The classes $O$ and $D_1$ (of acyclic graphs) are the most important objects of our interest in the whole paper.

The main results of this work estimate the $P$-sum-choice-number of a graph in terms of the $O$-sum-choice-number of another graph (Theorem 12) and in terms of hypergraph theory notions (Theorems 8, 10). These results are fruitful tools to obtain some new general results on the $P$-sum-choice-number (Corollaries 13, 14) and next the $D_1$-sum-choice-number (Corollaries 15, 16, 17, 18, 19, and Theorem 29). In particular we find the exact values of the $D_1$-sum-choice-numbers of all complete graphs and complete bipartite graphs $K_{p,q}$ where $q \geq \left\lfloor \frac{p+2}{2} \right\rfloor - 2$ or $p \leq 5$ (Theorems 23, 26, 27, 31).

Let $H = (V, E)$ be a hypergraph with vertex set $V$ and edge set $E$. A hypergraph $H' = (V', E')$ is called a subhypergraph of a hypergraph $H = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If additionally

- $E' = \{E \in E : E \subseteq V'\}$, then $H'$ is a subhypergraph of $H$ induced by $V'$ and it is denoted by $H[V']$,
- $V' = \bigcup_{E \in E'} E$, then $H'$ is a subhypergraph of $H$ induced by $E'$ and it is denoted by $H[E']$.

Let $H$ be a hypergraph with a vertex $x$. A star $H(x)$ in $H$ with a central vertex $x$ is a subhypergraph of $H$ induced by a set of all the edges containing $x$. By a $\beta$-star in a hypergraph $H$ with a central vertex $x$ we mean an arbitrary subhypergraph $H^\beta(x)$ of a hypergraph $H(x)$ such that both of the following conditions hold:

1. if $E$ is an edge of $H^\beta(x)$, then $|E| \geq 2$; and
2. if $E_1, E_2$ are two different edges of $H^\beta(x)$, then $E_1 \cap E_2 = \{x\}$.

The largest number of the edges of a $\beta$-star with a central vertex $x$ in a hypergraph $H$ we call the $\beta$-degree of a vertex $x$ in $H$ and denote $\deg_\beta^H(x)$. Note that each graph is a hypergraph, which allows us to use these definitions in the graph theory, too.

Let $G$ be a graph and $P$ be an induced hereditary class of graphs. By $F_\text{P}(G)$ we denote the set of all subsets $A$ of $V(G)$ such that $G[A] \in F(P)$. In many cases we consider a hypergraph associated with $G$ whose vertex set is $V(G)$ and edge set is $F_\text{P}(G)$. We call it the hypergraph of $G$ with respect to $P$.

## 2. GENERAL SUM-LIST-COLOURING

At the beginning of this section we formulate a few simple observations.

**Proposition 1.** If $P$ is an induced hereditary class of graphs and $G_1, G_2$ are graphs such that $G_1 \leq G_2$ and $G_2$ is $(f, \mathcal{P})$-choosable, then $G_1$ is $(f|_{V(G_1)}, \mathcal{P})$-choosable.
Proof. Observe that if \( c \) is an \((L, \mathcal{P})\)-colouring of \( G_2 \), then \( c' = c|_{V(G_1)} \) is the \((L', \mathcal{P})\)-colouring of \( G_1 \), where for \( L = \{L(v)\}_{v \in V(G_2)} \) the list assignment \( L' \) is represented by \( \{L(v)\}_{v \in V(G_1)} \).

Suppose, for a contradiction, that \( G_2 \) is \((f, \mathcal{P})\)-choosable and \( G_1 \) is not \((f|_{V(G_1)}, \mathcal{P})\)-choosable. It follows that there is a list assignment \( L_1 = \{L_1(v)\}_{v \in V(G_1)} \) such that \( |L_1(v)| = f|_{V(G_1)}(v) = f(v) \) for each \( v \in V(G_1) \) and \( G_1 \) is not \((L_1, \mathcal{P})\)-choosable. Let \( L_2 = \{L_2(v)\}_{v \in V(G_2)} \) be the list assignment of \( G_2 \) satisfying \( L_2(v) = L_1(v) \) for each \( v \in V(G_1) \) and \( L_2(v) = \{1, \ldots, f(v)\} \) for \( v \in V(G_2) \setminus V(G_1) \). Clearly, \( G_2 \) is \((L_1, \mathcal{P})\)-choosable since \( |L_2(v)| = f(v) \) for each \( v \in V(G_2) \). It implies the existence of \((L_2, \mathcal{P})\)-colouring \( c \) of \( G_2 \). Thus the first statement of the proof implies that \( c|_{V(G_1)} \) is the \((L_1, \mathcal{P})\)-colouring of \( G_1 \), contrary to our assumption.

**Proposition 2.** If \( P_1, P_2 \) are induced hereditary classes of graphs such that \( P_1 \subseteq P_2 \) and \( G \) is a graph, then
\[
\chi^P_{sc}(G) \geq \chi^P_{sc}(G).
\]

Proof. Observe that each \((L, P_1)\)-colouring of \( G \) is at the same time \((L, P_2)\)-colouring of \( G \). □

The next three statements present some properties of \( \mathcal{P}\)-sum-choice-numbers.

**Proposition 3.** If \( \mathcal{P} \) is an induced hereditary class of graphs and \( G_1, G_2, G_3 \) are any graphs satisfying \( G_2 \leq G_1, G_3 \leq G_1 \) and \( V(G_2) \cap V(G_3) = \emptyset \), then
\[
\chi^P_{sc}(G_1) \geq \chi^P_{sc}(G_2) + \chi^P_{sc}(G_3) + |V(G_1)| - |V(G_2)| - |V(G_3)|.
\]

Moreover, if \( V(G_2) \cup V(G_3) = V(G_1) \) and there is no edge \( A \) of the hypergraph of \( G_1 \) with respect to \( \mathcal{P} \) such that \( A \cap V(G_2) \neq \emptyset \) and \( A \cap V(G_3) \neq \emptyset \), then
\[
\chi^P_{sc}(G_1) = \chi^P_{sc}(G_2) + \chi^P_{sc}(G_3).
\]

Proof. Let \( \chi^P_{sc}(G_1) = m \). There exists a mapping \( f : V(G_1) \to \mathbb{N} \) such that \( G_1 \) is \((f, \mathcal{P})\)-choosable and \( \sum_{v \in V(G_1)} f(v) = m \). Let us consider mappings \( f_i : V(G_i) \to \mathbb{N} \) with \( i \in \{2, 3\} \) such that \( f_i(v) = f(v) \) for each \( v \in V(G_i) \). Evidently
\[
\sum_{v \in V(G_2)} f_2(v) + \sum_{v \in V(G_3)} f_3(v) + \sum_{v \in V(G_1) \setminus (V(G_2) \cup V(G_3))} f(v) = \sum_{v \in V(G_1)} f(v) = m.
\]

Fix \( i \in \{2, 3\} \). Since \( \mathcal{P} \) is an induced hereditary class of graphs and \( G_i \leq G_1 \), by Proposition 1, we have that \( G_i \) is \((f_i, \mathcal{P})\)-choosable and consequently
\[
\sum_{v \in V(G_i)} f_i(v) \geq \chi^P_{sc}(G_i).
\]

Thus the first statement of the theorem follows by an obvious inequality
\[
\sum_{v \in V(G_1) \setminus (V(G_2) \cup V(G_3))} f(v) \geq |V(G_1)| - |V(G_2)| - |V(G_3)|.
\]
To obtain the second statement, suppose that for a fixed \( i \in \{2, 3\} \) a mapping \( f_i : V(G_i) \rightarrow \mathbb{N} \) is such that \( G_i \) is \((f_i, \mathcal{P})\)-choosable and \( \sum_{v \in V(G_i)} f_i(v) = \chi_{sc}^{\mathcal{P}}(G_i) \).

Now let \( f : V(G_1) \rightarrow \mathbb{N} \) be defined as \( f(v) = f_2(v) \) if \( v \in V(G_2) \) and \( f(v) = f_3(v) \) if \( v \in V(G_3) \). Note that \( G_1 \) is \((f, \mathcal{P})\)-choosable. Indeed, if we put together any two \( \mathcal{P} \) sum-list-colourings of \( G_2 \) and \( G_3 \) from list assignments with the sizes specified by \( f_2 \) and \( f_3 \), respectively, then we obtain \( \mathcal{P} \) sum-list-colouring of \( G_1 \) (no edge of the hypergraph of \( G_1 \) with respect to \( \mathcal{P} \) has nonempty intersection with both \( V(G_2) \) and \( V(G_3) \)). Hence

\[
\chi_{sc}^{\mathcal{P}}(G_1) \leq \chi_{sc}^{\mathcal{P}}(G_2) + \chi_{sc}^{\mathcal{P}}(G_3).
\]

Taking into account the previous inequality, the statement equality is established which completes the proof.

Recall that an additive induced hereditary class of graphs is closed under taking disjoint union of graphs. Hence, applying Proposition 3, we conclude the following fact.

**Corollary 4.** If \( \mathcal{P} \) is an additive induced hereditary class of graphs and \( G_1, G_2 \) are disjoint graphs, then

\[
\chi_{sc}^{\mathcal{P}}(G_1 \cup G_2) = \chi_{sc}^{\mathcal{P}}(G_1) + \chi_{sc}^{\mathcal{P}}(G_2).
\]

Let \( k \in \mathbb{N} \). By a \( k \)-core \( G_k \) of a graph \( G \) we mean an induced subgraph of \( G \) obtained by successive pruning away of all the vertices of degree less than \( k \).

**Corollary 5.** If \( \mathcal{P} \) is an induced hereditary class of graphs and \( G \) is a graph, then

\[
\chi_{sc}^{\mathcal{P}}(G) = \chi_{sc}^{\mathcal{P}}(G_{\delta(\mathcal{P})}) + |V(G)| - |V(G_{\delta(\mathcal{P})})|.
\]

**Proof.** Observe that there is no edge of the hypergraph of \( G \) with respect to \( \mathcal{P} \) that has nonempty intersection with \( V(G) \setminus V(G_{\delta(\mathcal{P})}) \). Indeed, if such an edge exists, then the definitions of \( G_{\delta(\mathcal{P})} \) and the hypergraph of \( G \) with respect to \( \mathcal{P} \) force that vertices of this edge induce a graph \( F \) from \( \mathcal{F}(\mathcal{P}) \) in \( G \) such that \( \delta(F) < \delta(\mathcal{P}) \), which is impossible. It means that we can use the second statement of Proposition 3 to observe that

\[
\chi_{sc}^{\mathcal{P}}(G) = \chi_{sc}^{\mathcal{P}}(G_{\delta(\mathcal{P})}) + \chi_{sc}^{\mathcal{P}}(G[V(G) \setminus V(G_{\delta(\mathcal{P})})]).
\]

Additionally, our first observation implies \( G[V(G) \setminus V(G_{\delta(\mathcal{P})})] \in \mathcal{P} \), which yields \( \chi_{sc}^{\mathcal{P}}(G[V(G) \setminus V(G_{\delta(\mathcal{P})})]) = |V(G[V(G) \setminus V(G_{\delta(\mathcal{P})})])| \), by the definition of the \( \mathcal{P} \)-sum-choice-number of a graph.

Assuming that \( G_3 \) is empty we obtain one more straightforward consequence of Proposition 3.

**Corollary 6.** If \( \mathcal{P} \) is an induced hereditary class of graphs and \( G_1, G_2 \) are graphs such that \( G_2 \) is an induced subgraph of \( G_1 \), then

\[
\chi_{sc}^{\mathcal{P}}(G_1) \geq \chi_{sc}^{\mathcal{P}}(G_2) + |V(G_1)| - |V(G_2)|.
\]
It has to be mentioned here that many induced hereditary classes of graphs are at the same time \textit{hereditary}, which means closed under taking subgraphs (not necessarily induced ones). For example classes $\mathcal{O}$, $\mathcal{D}_1$ of edgeless and acyclic graphs, respectively, have such a property, unlike class $\mathcal{K}$ of graphs whose all connected components are complete. For hereditary classes of graphs the statement of Proposition 3 may by formulated in a stronger way. The proof of this new result is almost the same, except for the referring to the subgraph relation instead of the induced subgraph relation, when we decide about $(f, \mathcal{P})$-choosability of $G_i$ for $i \in \{2, 3\}$. We present an important consequence of this consideration.

\textbf{Corollary 7.} If $\mathcal{P}$ is a hereditary class of graphs and $G_1, G_2$ are graphs such that $G_2$ is a subgraph of $G_1$, then

$$\chi_{ac}^P(G_1) \geq \chi_{ac}^P(G_2) + |V(G_1)| - |V(G_2)|.$$  

Note that each hereditary class of graphs is induced hereditary but the opposite implication does not need to be fulfilled. Similarly, each induced subgraph of a graph is its subgraph but these notions are not equivalent. Hence Corollaries 6, 7 do not imply each other, despite the fact that their proofs are almost identical.

Now our purpose is the estimation of the $\mathcal{P}$-sum-choice-number of a graph.

\textbf{Theorem 8.} Let $\mathcal{P}$ be an induced hereditary class of graphs and $G$ be a graph. If $\mathcal{H}$ is the hypergraph of $G$ with respect to $\mathcal{P}$ and $v_1, \ldots, v_n$ is an arbitrary ordering of $V(G)$, then

$$\chi_{ac}^P(G) \leq \sum_{i=1}^{n} deg_{\mathcal{H}_i}^3(v_i) + n,$$

where $\mathcal{H}_i = \mathcal{H}[[v_1, \ldots, v_i]]$.

\textbf{Proof.} Given the assumed ordering $v_1, \ldots, v_n$ let $f(v_i) = deg_{\mathcal{H}_i}^3(v_i) + 1$. To finish the proof we shall show that $G$ is $(f, \mathcal{P})$-choosable. Let $L = \{L(v_i)\}_{i=1}^n$ be a list assignment such that $|L(v_i)| = f(v_i)$ for each $i \in \{1, \ldots, n\}$. We colour the vertices of $G$ greedily, in accordance with the ordering $v_1, \ldots, v_n$. Namely, in the $i$th step we assign the least colour from $L(v_i)$ to the vertex $v_i$ such that for each $a \in N$ the graph induced by the vertices coloured $a$ in the graph $G[[v_1, \ldots, v_i]]$ belongs to $\mathcal{P}$. If such a colouring exists for each step $i \in \{1, \ldots, n\}$, then the statement of the theorem holds.

Assume, for a contradiction, that $j$ is the first index such that in the $j$th step the requirement greedy colouring does not exist. Obviously $j > 1$ because $f(v_1) = 1$ and $G[[v_1]] = K_1 \in \mathcal{P}$, which means that one colour is enough to colour $v_1$. Thus $G[[v_1, \ldots, v_{j-1}]]$ has an $(L', \mathcal{P})$-colouring $c'$ for $L' = \{L(v_i)\}_{i=1}^{j-1}$. Moreover, no colour $b \in L(v_j)$ can be used to construct an $(L'', \mathcal{P})$-colouring $c''$ of $G[[v_1, \ldots, v_j]]$ from the list assignment $L'' = \{L(v_i)\}_{i=1}^{j-1}$ such that $c''(v_i) = c'(v_i)$ for $i \in \{1, \ldots, j-1\}$ and $c''(v_j) = b$. Since $\mathcal{H}_{j-1}$ has no monochromatic edge in $c'$ it follows that for each colour $b$ among $f(v_j) = deg_{\mathcal{H}_{j}}^3(v_j) + 1$ colours there exists an edge in $\mathcal{H}_j$ which would be monochromatic if we assign $b$ to $v_j$. Hence $deg_{\mathcal{H}_{j}}^3(v_j) \geq f(v_j) = deg_{\mathcal{H}_{j}}^3(v_j) + 1$, a contradiction.
Corollary 9. Let $\mathcal{P}$ be an induced hereditary class of graphs and $G$ be a graph. If $v_1, \ldots, v_n$ is an arbitrary ordering of $V(G)$, then

$$\chi_{sc}^\mathcal{P}(G) \leq \sum_{i=1}^{n} \left\lfloor \frac{\deg_G(v_i)}{\delta(P)} \right\rfloor + n \leq \frac{|E(G)|}{\delta(P)} + n,$$

where $G_i = G[v_1, \ldots, v_i]$.

Proof. Let $\mathcal{H}$ be the hypergraph of $G$ with respect to $\mathcal{P}$ and $\mathcal{H}_i = \mathcal{H}[v_1, \ldots, v_i]$. The assertion follows immediately by

- inequalities $\deg_{\mathcal{H}_i}(v_i) \leq \left\lfloor \frac{\deg_G(v_i)}{\delta(P)} \right\rfloor$ for each $i \in \{1, \ldots, n\}$; and
- equality $\sum_{i=1}^{n} \deg_{G_i}(v_i) = |E(G)|$,

which completes the proof.

Now we give a lower bound on the $\mathcal{P}$-sum-choice-number of a graph.

Theorem 10. Let $\mathcal{P}$ be an induced hereditary class of graphs and $G$ be an $n$-vertex graph. If $x$ is an arbitrary vertex of $G$ and $\mathcal{H}$ is the hypergraph of $G$ with respect to $\mathcal{P}$, then

$$\chi_{sc}^\mathcal{P}(G) \geq \deg_{\mathcal{H}}(x) + n.$$

Proof. Let $\deg_{\mathcal{H}}(x) = h$. Assume, for a contradiction, that $\chi_{sc}^\mathcal{P}(G) \leq h + n - 1$. Hence there exists a size function $f : V(G) \to \mathbb{N}$ such that $G$ is $(f, \mathcal{P})$-choosable and $\sum_{v \in V(G)} f(v) \leq h + n - 1$. First, we observe the following fact.

Claim 1. There exist $f(x)$ edges of $\mathcal{H}$, say $E_1, \ldots, E_{f(x)}$, each of which contains $x$ and whose all vertices $u$, except $x$, satisfy $f(u) = 1$. Moreover, for these edges and for $i \neq j$ the condition $(E_i \setminus \{x\}) \cap (E_j \setminus \{x\}) = \emptyset$ holds.

Otherwise, since $\deg_{\mathcal{H}}(x) = h$ it is at least $h - f(x) + 1$ edges of some $\mathcal{H}^\mathcal{P}$, say $E_1', \ldots, E_{h-f(x)+1}'$, such that each $E_k'$ contains at least one vertex different from $x$, say $u_k'$, satisfying $f(u_k') \geq 2$. Thus

$$\sum_{v \in V(G)} f(v) \geq 2(h - f(x) + 1) + f(x) + n - (h - f(x) + 2) = h + n.$$

It gives a contradiction to the assumption $\sum_{v \in V(G)} f(v) \leq h + n - 1$ and yields the claim.

Now we construct a list assignment $L = \{L(v)\}_{v \in V(G)}$ with the property $|L(v)| = f(v)$ for each $v \in V(G)$ such that $G$ is not $(L, \mathcal{P})$-choosable. Let
\(E_1, \ldots, E_{f(x)}\) be edges whose existence is confirmed by Claim 1. Next let \(L(u) = \{i\}\) for \(u \in E_1 \setminus \{x\}\) with \(i \in \{1, \ldots, f(x)\}\) and \(L(x) = \{1, \ldots, f(x)\}\). It is very easy to observe that \(G\) is not \((L, \mathcal{P})\)-choosable, which leads to \(\chi_{sc}^P(G) \geq h + n\).

**Corollary 11.** Let \(\mathcal{P}\) be an induced hereditary class of graphs and \(G\) be an \(n\)-vertex graph. If \(\mathcal{H}\) is the hypergraph of \(G\) with respect to \(\mathcal{P}\) and \(\mathcal{H}\) has a vertex \(x\) that is contained in each edge of \(\mathcal{H}\), then

\[
\chi_{sc}^P(G) = \deg_H^\beta(x) + n.
\]

In particular \(\chi_{sc}^P(G) = n\) if and only if \(G \in \mathcal{P}\).

**Proof.** For an ordering \(v_1, \ldots, v_n = x\) of the set \(V(G)\) we have \(\chi_{sc}^P(G) \leq \deg_H^\beta(x) + n\) by Theorem 8. Indeed, in this case \(\deg_H^\beta(v_i) = 0\) for all \(i \in \{1, \ldots, n - 1\}\). Theorem 10 implies the opposite inequality. In particular, it proves that if \(G \in \mathcal{P}\), then \(\chi_{sc}^P(G) = n\). Now it is enough to see that \(\chi_{sc}^P(G) = n\) implies \(G \in \mathcal{P}\). It is obvious, by the application of Theorem 10 in which each vertex can play a role of \(x\).

**2.1. Connection between \(\chi_{sc}^P\) and \(\chi_{sc}\)**

In this section we show the upper bound on the \(\mathcal{P}\)-sum-choice-number of a given graph \(G\) in terms of an \(\mathcal{O}\)-sum-choice-number of another graph that is constructed from \(G\) according to \(\mathcal{P}\). To do it we recall a useful notion from the hypergraph theory.

Let \(\mathcal{H}\) be a hypergraph. A **host graph** of a hypergraph \(\mathcal{H}\) is an arbitrary graph \(H\) whose vertex set is a subset of vertex set of \(\mathcal{H}\) and such that for every edge \(E\) of \(\mathcal{H}\) there exists an edge \(xy\) of \(H\) satisfying \(\{x, y\} \subseteq E\).

**Theorem 12.** If \(\mathcal{P}\) is an induced hereditary class of graphs, \(G\) is a graph and \(H\) is a host graph of the hypergraph of \(G\) with respect to \(\mathcal{P}\), then

\[
\chi_{sc}^P(G) \leq \chi_{sc}(H) + |V(G)| - |V(H)|.
\]

**Proof.** Let \(f : V(G) \to \mathbb{N}\) be an arbitrary mapping such that \(H\) is \((f|_{V(H)}, \mathcal{O})\)-choosable, \(\sum_{v \in V(H)} f(v) = \chi_{sc}(H)\) and \(f(v) = 1\) for \(v \in V(G) \setminus V(H)\). Thus

\[
\sum_{v \in V(G)} f(v) = \sum_{v \in V(H)} f(v) + \sum_{v \in V(G) \setminus V(H)} f(v) = \chi_{sc}(H) + |V(G)| - |V(H)|.
\]

We show that \(G\) is \((f, \mathcal{P})\)-choosable. For each list assignment \(L = \{L(v)\}_{v \in V(G)}\) satisfying \(|L(v)| = f(v)\) when \(v \in V(G)\) we consider a list assignment \(L' = \{L(v)\}_{v \in V(H)}\). Obviously \(H\) is \((L', \mathcal{O})\)-choosable. It follows that there exists a colouring \(c : V(H) \to \mathbb{N}\) with the property that for any two adjacent vertices \(v_1, v_2\) of \(H\) we obtain \(c(v_1) \neq c(v_2)\) and \(c(v) \in L(v)\) for \(v \in V(H)\). The construction of \(H\) guarantees that for each \(A \subseteq V(G)\) such that \(G[A] \in \mathcal{F}(\mathcal{P})\) there
are two vertices \( v_1, v_2 \) satisfying \( c(v_1) \neq c(v_2) \) and \( \{v_1, v_2\} \subseteq A \). We extend \( c \) to \( c' : V(G) \to N \) so that \( c'(v) = c(v) \) for \( v \in V(H) \) and \( c'(v) = a \), where \( L(v) = \{a\} \) for \( v \in V(G) \setminus V(H) \). Evidently \( c' \) forces the \((L, \mathcal{P})\)-choosability of \( G \). Hence \( G \) is \((f, \mathcal{P})\)-choosable, which implies the assertion.

Referring to Theorem 12 it has to be mentioned that the edges of a host graph \( H \) are not necessarily the edges of a graph \( G \). We assume the inclusion between these two sets of edges in the first of the two next facts, unlike the second one.

**Corollary 13.** If \( \mathcal{P} \) is an induced hereditary class of graphs, \( G \) is a graph and \((E_1, E_2)\) is a partition of the edge set of \( G \) such that \( G[E_1] \in \mathcal{P} \), then

\[
\chi_{sc}^{\mathcal{P}}(G) \leq \chi_{sc}(G[E_2]) + |V(G)| - |V(G[E_2])|.
\]

**Proof.** Observe that \( G[E_2] \) is a host graph of the hypergraph of \( G \) with respect to \( \mathcal{P} \). The conclusion follows by Theorem 12.

**Corollary 14.** Let \( \mathcal{P} \) be an induced hereditary class of graphs, \( G \) be a graph, \( B \subseteq V(G) \) and \( B \) be a disjoint union of sets \( B_1, \ldots, B_t \) with \( t \in \mathbb{N} \). If for each \( A \in \mathcal{F}_\mathcal{P}(G) \) there exists \( i \in \{1, \ldots, t\} \) such that \( |A \cap B_i| \geq 2 \), then

\[
\chi_{sc}^{\mathcal{P}}(G) \leq \sum_{i=1}^{t} \left( \frac{|B_i| + 1}{2} \right) + |V(G)| - \sum_{i=1}^{t} |B_i|.
\]

**Proof.** Let \( H \) be the union of \( K_{|B_i|} \) for \( i \in \{1, \ldots, t\} \) with \( V(H) = \bigcup_{i=1}^{t} B_i \). Clearly \( H \) is a host graph of the hypergraph of \( G \) with respect to \( \mathcal{P} \). It is known (\([12]\)) that

\[
\chi_{sc}(K_{|B_i|}) = \left( \frac{|B_i| + 1}{2} \right).
\]

The assertion follows by the application of Corollary 4 to the additive class \( \mathcal{O} \) and by Theorem 12.

3. ACYCLIC SUM-LIST-COLOURINGS

In this section we focus our attention on class \( D_1 \) of all the acyclic graphs. Applying the results on an arbitrary induced hereditary class of graphs that were presented in the previous section, we prove some properties of the \( D_1 \)-sum-choice-number of a graph. First we formulate some new facts based on Theorem 12. Here \( c(G) \) denotes the number of all the connected components of a graph \( G \).

**Corollary 15.** For every graph \( G \) it holds

\[
\chi_{sc}^{\mathcal{D}}(G) \leq |E(G)| + c(G).
\]

**Proof.** Let \( \mu(G) = |E(G)| - |V(G)| + c(G) \) (in the literature \( \mu(G) \) is called the cyclomatic number of a graph \( G \)). It is well known that \( \mu(G) \geq 0 \) and there exists
an edge set \( M \) in \( G \) of cardinality \( \mu(G) \) such that the graph resulting from \( G \) by the removal of \( M \) is acyclic. Let \( H \) be the graph \( G[M] \) induced in \( G \) by the edge set \( M \). Thus, by Corollary 13,

\[
\chi_{sc}^{D_1}(G) \leq \chi_{sc}(H) + |V(G)| - |V(H)|.
\]

(Note that if \( \mu(G) = 0 \), then \( M = \emptyset \) and \( \chi_{sc}(H) = 0 \)).

Since for every graph \( H' \) it holds \( \chi_{sc}(H') \leq |E(H')| + |V(H')| \) ([12]) we have

\[
\chi_{sc}^{D_1}(G) \leq |E(H)| + |V(G)|.
\]

But \( |E(H)| = \mu(G) = |E(G)| - |V(G)| + c(G) \), which gives

\[
\chi_{sc}^{D_1}(G) \leq |E(G)| + c(G).
\]

□

The following two results give some upper bounds on \( D_1 \)-sum-choice-numbers of special classes of graphs. They are the straightforward consequences of Corollary 15.

**Corollary 16.** For every \( n \)-vertex planar graph \( G \) such that each connected component of \( G \) contains at least one cycle it holds

\[
\chi_{sc}^{D_1}(G) \leq \min \left\{ 3n - 5c(G), \frac{5}{2}n - 3c(G) \right\}.
\]

**Proof.** Let \( G_1, \ldots, G_{c(G)} \) be the connected components of \( G \). Since for each \( i \in \{1, \ldots, c(G)\} \) the graph \( G_i \) is planar and has at least one cycle we obtain \( |E(G_i)| \leq 3|V(G_i)| - 6 \), using the famous Euler Formula. It gives \( |E(G)| \leq 3n - 6c(G) \). Since \( \delta(D_1) = 2 \) it implies the assertion, by Corollaries 9 and 15.

**Corollary 17.** If \( G \) is an \( n \)-vertex graph such that each connected component of \( G \) is \( k \)-degenerate and has at least \( k + 1 \) vertices, then

\[
\chi_{sc}^{D_1}(G) \leq kn + \left( 1 - \frac{c(G)}{2k} - 1 \right)c(G).
\]

**Proof.** Let \( G_1, \ldots, G_{c(G)} \) be connected components of \( G \). By the known formula on the maximum number of edges of an arbitrary \( k \)-degenerate graph that has at least \( k + 1 \) vertices we have

\[
|E(G_i)| \leq k \left( |V(G_i)| - \frac{1}{2}(k + 1) \right),
\]

where \( i \in \{1, \ldots, c(G)\} \). The conclusion follows by Corollary 15.

**Corollary 18.** If \( G \) is an \( n \)-vertex planar bipartite graph, then

\[
\chi_{sc}^{D_1}(G) \leq 2n - 1.
\]
Proof. It is known that the edge set of each planar bipartite graph can be partitioned into two parts, each of which induces a forest ([15]). Since for any tree \( T \) it holds \( \chi_{sc}(T) = 2|V(T)| - 1 \) (it was first observed by Mike Albertson in 2001 and mentioned in [12] as stated privately), Corollaries 4 and 13 give the conclusion.

The girth of a graph \( G \) is the length of the shortest cycle in \( G \). In [16] it was proved that the edge set of each planar graph \( G \) whose girth satisfies \( g(G) \geq 8 \) can be partitioned into two sets such that one of them induces a forest and the second one is a matching of \( G \). Since \( \chi_{sc}(mK_2) = 3m \) for any \( m \in \mathbb{N} \) and because of Corollary 13 we can state the following result.

Corollary 19. If \( G \) is an \( n \)-vertex planar graph with \( g(G) \geq 8 \), then
\[
\chi_{D1}^{sc}(G) \leq \frac{3n}{2}.
\]

3.1. Bipartite graphs

In this section we analyse acyclic sum-list-colouring of complete bipartite graphs and of bipartite graphs in general. These classes of graphs seem to be important to develop some of the topics related to the subject of this paper. In [6] Erdős et al. examined list colourings of complete bipartite balanced graphs to observe the difference between the concepts of proper colouring and list colouring. Next it was proved that the choice number (the minimum number \( d \in \mathbb{N} \) such that for the constant function \( f \equiv d \) a graph is \((f,O)\)-choosable) depends on an average degree of a graph [1] unlike the sum-choice-number ([O-sum-choice-number]) [7] and the last observation was based on the behaviour of the sum-choice-number of the unbalanced complete bipartite graphs. Some exact values of the sum-choice-number of \( K_{p,q} \), but only for \( p \in \{1,2,3\} \) were given ([2, 9]). We hope that our consideration connected to complete bipartite graphs will expand the knowledge from that scope.

At the beginning consider two upper bounds on \( \chi_{D1}^{sc}(K_{p,q}) \) obtained by the application of Corollaries 15 and 14, respectively. Namely, from the first one we have \( \chi_{D1}^{sc}(K_{p,q}) \leq pq + 1 \) and from the second one \( \chi_{D1}^{sc}(K_{p,q}) \leq \left\lceil \frac{p+1}{2} \right\rceil + q \) if we take \( B \) equal to the partite set of \( K_{p,q} \) with the cardinality \( p \). In the next part of this section, we show some better estimates for \( \chi_{D1}^{sc}(K_{p,q}) \). In many cases we give the exact values of the analysed invariant. Let us start with the following identity.

Lemma 20. If \( p, r \in \mathbb{N} \) and \( r \leq p - 1 \), then
\[
p + \binom{r+1}{2} + r(p-r-1) = \binom{p+1}{2} - \binom{p-r}{2}.
\]

Proof. First we establish combinatorially the auxiliary identity \( \binom{r}{2} + r(p-r) = \binom{p}{2} - \binom{p-r}{2} \). It is easy to see that both sides of this equation count the number
Theorem 21 was obtained immediately from Theorem 8 with special order.

Theorem 8 implies

$$H = P$$

which completes the proof.

Proof. Let $$(v_1, v_2, \ldots, v_p)$$ be the vertex set of this auxiliary identity we obtain the conclusion. So, the sides are equal to each other. Now we observe that adding $p$ to both sides of this auxiliary identity we obtain the conclusion.

In what follows, we adopt the convention $\binom{n}{m} = 0$ when $n < m$.

**Theorem 21.** If $p, q \in \mathbb{N}$ and $q \geq p$, then

$$p + q + \min\left\{ p - 1, \left\lfloor \frac{q}{2} \right\rfloor \right\} \leq \chi_{sc}^{D_1}(K_{p,q}) \leq \left( \frac{p+1}{2} \right) + q - \left( \frac{p-1}{2} \right).$$

**Proof.** Let $$(P, Q)$$ be the bipartition of $V(K_{p,q})$ such that $Q = \{v_1, \ldots, v_q\}$ and $P = \{v_{q+1}, \ldots, v_{q+r}\}$. Next let $H$ be the hypergraph of $K_{p,q}$ with respect to $D_1$ and $H_t = \mathcal{H}[\{v_1, \ldots, v_t\}]$. Note that $\deg_H^*(v_i) = 0$ for $i \in \{1, \ldots, q\}$ and $\deg_H^*(v_i) = \min\left\{ i - q + 1, 1 \right\}$ for $i \in \{q + 1, \ldots, q + p\}$. Indeed, the maximum number of edges of a $\beta$-star of $H_t$ with a central vertex $v_i \in P$ is bounded from above by the number of disjoint pairs of vertices in $Q$ and by the number of pairs that the vertex $v_i$ create with different vertices in $\{v_{q+1}, \ldots, v_{q+r}\}$. The smaller of these two numbers equals $\deg_H^*(v_i)$. Thus Theorem 10 applied to the vertex $v_{q+r}$ leads to

$$p + q + \min\left\{ p - 1, \left\lfloor \frac{q}{2} \right\rfloor \right\} \leq \chi_{sc}^{D_1}(K_{p,q}).$$

Theorem 8 implies

$$\chi_{sc}^{D_1}(K_{p,q}) \leq q + p + \sum_{i=q+1}^{q+p} \min\left\{ i - q + 1, 1 \right\} = q + p + \sum_{j=1}^{p} \min\left\{ j - 1, 1 \right\}.$$

Next for $p - 1 \leq \left\lfloor \frac{q}{2} \right\rfloor$ it holds that

$$\chi_{sc}^{D_1}(K_{p,q}) \leq q + p + \sum_{j=1}^{p} (j - 1) = q + \left( \frac{p+1}{2} \right),$$

which was previously concluded by Corollary 14 and for $p - 1 > \left\lfloor \frac{q}{2} \right\rfloor$ we have

$$\chi_{sc}^{D_1}(K_{p,q}) \leq q + \sum_{j=1}^{\left\lfloor \frac{q}{2} \right\rfloor+1} (j - 1) + \sum_{j=\left\lfloor \frac{q}{2} \right\rfloor+2}^{p} \left\lfloor \frac{q}{2} \right\rfloor$$

$$= q + p - \left( \left\lfloor \frac{q}{2} \right\rfloor + 1 \right) - \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) \frac{q}{2} = \left( \frac{p+1}{2} \right) - \left( \frac{p-1}{2} \right) + q,$$

where the last equation follows by Lemma 20 used with parameters $p$ and $r = \left\lfloor \frac{q}{2} \right\rfloor$, which completes the proof.

It should be mentioned here that the upper bound on $\chi_{sc}^{D_1}(K_{p,q})$ given in Theorem 21 was obtained immediately from Theorem 8 with special ordering of
vertices of $K_{p,q}$. In the next part of the paper, we use other tools to compute the $D_1$-sum-choice-numbers of some complete bipartite graphs. Moreover, we show that the application of Theorem 8 with any ordering of the vertex set is not enough to establish these values.

First we present the result that supports obtaining exact values of the $D_1$-sum-choice-numbers of many graphs.

**Lemma 22.** Let $k, p, q \in \mathbb{N}$ and let $(P, Q)$ be the bipartition of $V(K_{p,q})$. Next let $P' \subseteq P$, $Q' \subseteq Q$ and $|P'| = k + 1$ and $|Q'| = 2k$. If $f : V(K_{p,q}) \to \mathbb{N}$ is a function such that $f(v) \leq k$ for $v \in P'$ and $f(v) = 1$ for $v \in Q'$, then $K_{p,q}$ is not $(f, D_1)$-choosable.

**Proof.** Without loss of generality suppose that $|P| = p$ and $P' = \{v_1, \ldots, v_{k+1}\}$, $Q' = \{v_{p+1}, \ldots, v_{2k}\}$ and $f(v) = k$ for $v \in P'$. Consider a list assignment $L = \{L(v_i)\}_{v_i \in P}$ such that $L(v_i) = \{1, \ldots, k\}$ for $i \in \{1, \ldots, k+1\}$ and $L(v_{p+1}) = \left\{\left\lfloor \frac{i}{2} \right\rfloor \right\}$ for $i \in \{1, \ldots, 2k\}$. We show that $K_{p,q}$ is not $(L, D_1)$-choosable. For a contradiction, suppose that $c$ is $(L, D_1)$-colouring of $K_{p,q}$. Since $|\bigcup_{i=1}^{k+1} L(v_i)| = k$, by the pigeonhole principle it follows that there exist two different indices $i_1, i_2 \in \{1, \ldots, k+1\}$ such that $c(v_{i_1}) = c(v_{i_2})$ and moreover the construction of $L$ forces $c(v_{i_1}) = j \in \{1, \ldots, k\}$. Note that the definition of $c$ implies $c(v_{p+2j-1}) = \left\lfloor \frac{2j-1}{2} \right\rfloor = c(v_{p+2j}) = \left\lfloor \frac{2j}{2} \right\rfloor = j$. Consequently $v_{i_1}, v_{i_2}, v_{p+2j-1}, v_{p+2j}$ induce a monochromatic cycle coloured $j$ in $K_{p,q}$, a contradiction.

**Theorem 23.** If $p, q \in \mathbb{N}$, $q \geq p$ and $q \geq \left(\frac{p+1}{2}\right) - 2$, then

$$\chi^{D_1}_{sc}(K_{p,q}) = \left(\frac{p+1}{2}\right) + q.$$

**Proof.** The inequality $\chi_{sc}^{D_1}(K_{p,q}) \leq \left(\frac{p+1}{2}\right) + q$ is obvious, by Theorem 21. We apply Corollary 11 to confirm the statement for $p \in \{1, 2\}$. Let us assume that $p \geq 3$. Suppose, for a contradiction, that $f : V(K_{p,q}) \to \mathbb{N}$ is a size function such that $K_{p,q}$ is $(f, D_1)$-choosable and $\sum_{v \in V(K_{p,q})} f(v) \leq \left(\frac{p+1}{2}\right) + q - 1$. We show that it yields the existence of at least $2(p - 1) + 1$ vertices for which the value of $f$ equals one. Otherwise

$$\sum_{v \in V(G)} f(v) \geq 2(p - 1) + 2(p + q - (2(p - 1))) = 2p - 2 + 2p + 2q - 4(p - 1) = 2q + 2.$$

Since $q \geq \left(\frac{p+1}{2}\right) - 2$ it follows

$$\sum_{v \in V(G)} f(v) \geq \left(\frac{p+1}{2}\right) - 2 + q + 2 = \left(\frac{p+1}{2}\right) + q.$$
giving a contradiction. Hence we have at least $2(p - 1) + 1 = 2p - 1$ vertices $v \in V(K_{p,q})$ satisfying $f(v) = 1$.

Now, let $(P, Q)$ be the bipartition of $V(K_{p,q})$, where $P = \{v_1, \ldots, v_p\}$ and $Q = \{v_{p+1}, \ldots, v_{p+q}\}$. Next let $C = \{v \in V(K_{p,q}) : f(v) = 1\}$. Note that, by the application of Lemma 22 with $k = 1$ we conclude that either $|C \cap P| \leq 1$ or $|C \cap Q| \leq 1$ and because of the restriction $|C| \geq 2(p - 1) + 1$ we have $|C \cap P| \leq 1$.

Next, without loss of generality (offer reordering of the set $P$), let, for $i \in \{1, \ldots, p\}$, it be $f(v_i) \leq f(v_\ell)$ when $i < \ell$. We shall show that for each permissible $i$ the value $f(v_i)$ must be greater than or equal to $i$. Otherwise (there is $i$ with $f(v_i) \leq i - 1$), we apply Lemma 22 with $k = i - 1$, $P' = \{v_1, \ldots, v_i\}$ and $Q' \subseteq C \cap Q$ of size $2(i - 1)$ to show that $K_{p,q}$ is not $(f, D_1)$-choosable, contrary to the assumption.

Hence $f(v_i) \geq i$ for $i \in \{1, \ldots, p\}$ and consequently
\[
\sum_{v \in V(G)} f(v) \geq 1 + \ldots + p + q = \left(\frac{p + 1}{2}\right) + q.
\]

which contradicts the assumption $\sum_{v \in V(G)} f(v) \leq \left(\frac{p + 1}{2}\right) + q - 1$. $\square$

Note that we know each value $\chi_{sc}^{D_1}(K_{p,q})$ for $q \geq \left(\frac{p + 1}{2}\right) - 2$. Hence it holds that $\lim_{q \to \infty} \frac{\chi_{sc}^{D_1}(K_{p,q})}{p + q} = 1$. A similar result, but with stronger assumptions, was obtained for $\chi_{sc}(K_{p,q})$ in [7]. This result states $\lim_{p \to \infty; q > p^2 \log p} \frac{\chi_{sc}(K_{p,q})}{p + q} = 2$.

In the proof of the next result we use the Hall Theorem, which for a given family of finite sets presents a necessary and sufficient condition for being able to select a distinct element from each set.

**Theorem 24.** [10] Let $n \in \mathbb{N}$ and $\{L_i\}_{i=1}^n$ be a collection of finite sets. There exists a selection of distinct elements $x_1, \ldots, x_n$ such that for each $i \in \{1, \ldots, n\}$ $x_i \in L_i$ if and only if for every $J \subseteq \{1, \ldots, n\}$ it holds $|\bigcup_{i \in J} L_i| \geq |J|$.

In what follows if $a, b \in \mathbb{N} \cup \{0\}$ and $b < a$, then the set $\{a, \ldots, b\}$ is empty.

**Lemma 25.** Let $k, p, q \in \mathbb{N}, k \leq q$ and let $(P, Q)$ be the bipartition of $V(K_{p,q})$ such that $Q = \{v_1, \ldots, v_q\}$ and $P = \{v_{q+1}, \ldots, v_{q+p}\}$. If $f : V(K_{p,q}) \to \mathbb{N}$ is a function such that:

1. $f(v_i) = 1$ for $i \in \{1, \ldots, k\}$ and
2. $f(v_{k+i}) = i + 1$ for $i \in \{1, \ldots, q - k\}$ and
3. $f(v_{q+i}) = i$ for $i \in \{1, \ldots, \min\left\{\left\lfloor \frac{k}{2}\right\rfloor, p\right\}$ and
4. \( f(v_{q+i}) = \left\lfloor \frac{k}{2} \right\rfloor + 1 \) for \( i \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor + 1, \ldots, p \right\} \),

then \( K_{p,q} \) is \((f, D_1)\)-choosable.

**Proof.** Let \( L = \{L(v_1)\}_{i=1}^{p+q} \) be a list assignment satisfying \(|L(v_i)| = f(v_i)\) for each \( i \in \{1, \ldots, p+q\} \). We construct \((L, D_1)\)-colouring \( c \) of \( K_{p,q} \), step by step, starting with \( c(v_1) \) and putting \( c(v_i) \) in the \( i^{th} \) step. Thus we colour the vertices of \( Q \) choosing in each step \( i \in \{1, \ldots, q\} \) one element from \( L(v_i) \) such that the multiset \( \{c(v_1), \ldots, c(v_i)\} \) has the smallest possible number of elements represented more than once. We denote this number by \( s_i \). Moreover, before we determine \( c(v_i) \) all the colours represented once in \( \{c(v_1), \ldots, c(v_{i-1})\} \) are called bad in this step. Now we shall observe that in \( i^{th} \) step, with \( i \in \{1, \ldots, q\} \), the number \( s_i \) does not exceed \( \left\lfloor \frac{k}{2} \right\rfloor \).

Indeed, for \( i \in \{1, \ldots, k\} \) we have no choice \((f(v_i) = 1)\), but the condition \( s_i \leq \left\lfloor \frac{k}{2} \right\rfloor \) is obviously fulfilled. So we decide about \( c(v_{k+j}) \) for some \( j \geq 1 \). If \( s_{k+j-1} < \left\lfloor \frac{k}{2} \right\rfloor \), then we choose a colour from \( L(v_{k+j}) \) to minimize the number \( s_{k+j} \) of colours represented more than once in \( \{c(v_1), \ldots, c(v_{k+j})\} \). Regardless of our choice the condition \( s_{k+j} \leq \left\lfloor \frac{k}{2} \right\rfloor \) holds. If \( s_{k+j-1} = \left\lfloor \frac{k}{2} \right\rfloor \), then the number of bad colours in this step is at most \( k+j-1-2\left\lfloor \frac{k}{2} \right\rfloor \) which is not greater than \( j \). Recall that \( f(v_{k+j}) = j+1 \). Therefore we have at least one colour in \( L(v_{k+j}) \) such that choosing it as \( c(v_{k+j}) \) the equality \( s_{k+j} = s_{k+j-1} \) is satisfied. Thus in both cases \( s_{k+j} \leq \left\lfloor \frac{k}{2} \right\rfloor \).

Now we apply Theorem 24 to construct the set \( \{c(v_{q+1}), \ldots, c(v_{q+\left\lfloor \frac{k}{2} \right\rfloor})\} \) consisting of distinct elements which is possible because for each \( r \) sets \( L(v_{q+i_1}), \ldots, L(v_{q+i_r}) \) we have \(|\bigcup_{i=1}^r L(v_{q+i_r})| \geq \max\{i_1, \ldots, i_r\} \geq r\). Finally, for \( i \in \left\{ \left\lfloor \frac{k}{2} \right\rfloor + 1, \ldots, p \right\} \), as \( c(v_{q+i}) \) we take a colour from \( L(v_{q+i}) \) which is not represented more than once in the multiset \( \{c(v_1), \ldots, c(v_q)\} \). It is always possible since \( s_q \leq \left\lfloor \frac{k}{2} \right\rfloor \). Note that each cycle in \( K_{p,q} \) must contain at least two vertices from \( P \) and at least two vertices from \( Q \). The construction of \( c \) guarantees that any colour appearing at least twice in \( Q \) appears at most once in \( P \). It forces \((L, D_1)\)-colourability of \( K_{p,q} \). 

Theorems 21, 23 are useful tools in establishing or estimating the \( D_1 \)-sum-choice-numbers of many complete bipartite graphs. The proofs of corresponding results frequently use Proposition 1.

**Theorem 26.** If \( q \in \mathbb{N} \), then

\[
\chi_{sc}^{D_1}(K_{1,q}) = q + 1, \quad \chi_{sc}^{D_1}(K_{2,q}) = q + 3, \quad \text{if } q \geq 2,
\]

\[
\chi_{ac}^{D_1}(K_{4,q}) = \begin{cases} 8, & \text{if } q = 3, \\ q + 6, & \text{if } q \geq 4, \end{cases}
\]

\[
\chi_{sc}^{D_1}(K_{4,q}) = \begin{cases} 12, & \text{if } q = 4, \\ 14, & \text{if } q = 5, \\ q + 10, & \text{if } q \geq 6. \end{cases}
\]
Proof. The first two equalities follow by Theorem 23. The same theorem implies that \( \chi^D_1(K_{3,3}) = q + 6 \) for \( q \geq 4 \). Moreover, Theorem 21 gives us \( 7 \leq \chi^D_1(K_{3,3}) \leq 8 \). Assume that \( f \) is a size function that realizes the equality \( \sum_{v \in V(K_{3,3})} f(v) = 7 \).

Observe that the assumptions of Lemma 22 with \( k = 1 \) have to be satisfied. It implies the \((f, D_1)\)-nonchoosability of \( K_{3,3} \) and leads to the conclusion \( \chi^D_1(K_{3,3}) = 8 \).

The application of Theorem 23 again yields \( \chi^D_1(K_{4,q}) = q + 10 \) for \( q \geq 8 \). We need decide about the values \( \chi^D_1(K_{4,q}) \) for \( q \in \{4, 5, 6, 7\} \). If for a size function \( f \) it holds \( \sum_{v \in V(K_{4,q})} f(v) \leq 11 \), then \( f \) satisfies the assumptions of Lemma 22 with \( k = 1 \) or with \( k = 2 \), which leads to \((f, D_1)\)-nonchoosability. Thus \( 12 \leq \chi^D_1(K_{4,4}) \).

Next we apply Lemma 25 with parameter \( k = 3 \) to confirm \( \chi^D_1(K_{4,4}) = 12 \).

Theorem 21 implies that the value \( \chi^D_1(K_{4,5}) \) is bounded from above by 14. Suppose, for a contradiction, that \( f : V(K_{4,5}) \to \mathbb{N} \) is a size function such that

\[
\sum_{v \in V(K_{4,5})} f(v) = 13 \quad \text{and} \quad K_{4,5} \text{ is } (f, D_1)\text{-choosable.}
\]

Observe that in the multiset \( \{f(v) : v \in V(K_{4,5})\} \) number one is repeated \( x \) times, where \( x \geq 5 \). If \( x \geq 7 \), then the assumptions of Lemma 22 with \( k = 1 \) are satisfied, giving a contradiction. If \( x \in \{5, 6\} \), then unavoidably the assumptions of Lemma 22 with \( k = 1 \) or \( k = 2 \) are satisfied, implying a contradiction in this case too. Hence \( \chi^D_1(K_{4,5}) = 14 \).

Applying Theorem 21 one can deduce \( \chi^D_1(K_{4,6}) \leq 16 \). If \( f : V(K_{4,6}) \to \mathbb{N} \) is a size function such that \( K_{4,6} \) is \((f, D_1)\)-choosable and \( \sum_{v \in V(K_{4,6})} f(v) = 15 \) and \( (P, Q) \) is the bipartition of \( V(K_{4,6}) \) such that \( |P| = 4 \), then for each vertex \( v \in Q \) we have \( f(v) = 1 \). Otherwise, for \( w \in Q \) that satisfies \( f(w) \geq 2 \) we obtain

\[
\sum_{v \in V(K_{4,6}) \setminus \{w\}} f(v) \leq 13,
\]

which leads to \( \chi^D_1(K_{4,5}) \leq 13 \) and contradicts the previously confirmed statement. Hence \( f(v) = 1 \) for each \( v \in Q \). If for all \( v \in P \) we have \( f(v) \leq 3 \), then the assumptions of Lemma 22 with \( k = 3 \) are satisfied, which yields the \((f, D_1)\)-nonchoosability of \( K_{4,6} \). If there is \( w \in P \) such that \( f(w) \geq 4 \), then

\[
\sum_{v \in V(K_{4,6}) \setminus \{w\}} f(v) \leq 11,
\]

giving \( \chi^D_1(K_{3,6}) \leq 11 \), which is impossible. Hence

\[
\chi^D_1(K_{4,6}) = 16.
\]

Finally, by the application of Corollary 6 with \( G_1 = K_{4,7} \) and \( G_2 = K_{4,6} \) we have \( 17 \leq \chi^D_1(K_{4,7}) \). The upper bound on \( \chi^D_1(K_{4,7}) \) stated in Theorem 21 gives the inequality \( \chi^D_1(K_{4,7}) \leq 17 \) and confirms the last assertion.

The reader has to avoid the conviction that there always exists an order on the set of vertices of a graph for which the upper bound given in Theorem 8 realizes the \( D_1 \)-sum-choice-number of this graph. To see it, suppose that \( H \) is the hypergraph of \( K_{4,4} \) with respect to \( D_1 \). Next assume that there is an ordering
\begin{align*}
(v_1, \ldots, v_8) \text{ of } V(K_{4,4}) \text{ for which } \sum_{i=1}^{8} \deg_{H_i}^\beta(v_i) = \chi_{sc}^{D_1}(K_{4,4}) = 12, \text{ where } H_i = H([v_1, \ldots, v_8]) \text{ for } i \in \{1, \ldots, 8\}. \text{ In this case, because } \deg_{H_i}^\beta(v_8) = 2, \text{ we have } \\
\sum_{i=1}^{7} \deg_{H_i}^\beta(v_i) + 7 = 9. \text{ On the other hand, Theorem 8 and Theorem 26 imply } \\
10 = \chi_{sc}^{D_1}(K_{3,4}) \leq \sum_{i=1}^{7} \deg_{H_i}^\beta(v_i) + 7, \text{ which confirms that the assumed ordering of } V(K_{4,4}) \text{ does not exist.}
\end{align*}

**Theorem 27.** If \( q \in \mathbb{N} \), then

\[
\chi_{sc}^{D_1}(K_{5,q}) = \begin{cases} 
17, & \text{if } q = 5 \\
19, & \text{if } q = 6 \\
21, & \text{if } q = 7 \\
q + 15, & \text{if } q \geq 8.
\end{cases}
\]

**Proof.** Theorem 21 leads to \( \chi_{sc}^{D_1}(K_{5,5}) \leq 17 \). Suppose that \( \chi_{sc}^{D_1}(K_{5,5}) \leq 16 \) and let \( f : V(K_{5,5}) \to \mathbb{N} \) be an arbitrary size function that realizes this assumption. If \( f(w) \geq 3 \) for at least one \( w \in V(K_{5,5}) \), then for a subgraph \( K_{4,5} \) obtained from \( K_{5,5} \) by the removal of the vertex \( w \) we have \( \sum_{v \in V(K_{4,5})} f(v) \leq 13 \). Because \( K_{4,5} \) is \((f|_{V(K_{5,5})\setminus\{w\}), D_1)\)-choosable by Proposition 1, it implies \( \chi_{sc}^{D_1}(K_{5,5}) \leq 13 \), which is impossible. Hence the multiset \( \{f(v) : v \in V(K_{5,5})\} \) consists only of values one and two. If \( f \) satisfies the assumptions of Lemma 22 with \( k = 1 \) or \( k = 2 \), then \( K_{5,5} \) is not \((f, D_1)\)-choosable. Otherwise (the assumptions of Lemma 22 are satisfied with neither \( k = 1 \) nor \( k = 2 \)) let \( (P, Q) \) be the bipartition of \( V(K_{5,5}) \) such that \( P = \{v_1, \ldots, v_5\} \) and \( Q = \{v_6, \ldots, v_{10}\} \). Without loss of generality the only case to consider is \( f(v_1) = f(v_2) = f(v_3) = f(v_6) = 1 \) and \( f(v) = 2 \) for the remaining vertices. Consider a list assignment \( L = \{L(v_i)\}_{i=1}^{10} \) such that \( L(v_1) = L(v_2) = L(v_6) = \{1\}, L(v_3) = \{2\}, L(v_4) = L(v_5) = \{2, 3\}, L(v_7) = L(v_8) = \{1, 2\} \) and \( L(v_9) = L(v_{10}) = \{1, 3\} \). We leave to the reader the confirmation of the fact that \( K_{5,5} \) is not \((L, D_1)\)-choosable implying the \((f, D_1)\)-nonchoosability. Hence \( \chi_{sc}^{D_1}(K_{5,5}) = 17 \).

Now by Corollary 6 applied to \( G_1 = K_{5,6} \) and \( G_2 = K_{5,5} \) we have \( \chi_{sc}^{D_1}(K_{5,6}) \geq 18 \). Moreover, using Lemma 25 with \( k = 3 \) we can see that \( \chi_{sc}^{D_1}(K_{5,6}) \leq 19 \). It remains to show that there exists no size function \( f : V(K_{5,6}) \to \mathbb{N} \) such that \( K_{5,6} \) is \((f, D_1)\)-choosable and \( \sum_{v \in V(K_{5,6})} f(v) = 18 \). For a contradiction, suppose that such an \( f \) exists and \((P, Q) \) is the bipartition of \( V(K_{5,6}) \) with \( |P| = 5 \). We can now proceed analogously to the previous proof to observe that \( f(v) = 1 \) for each \( v \in P \). There are at least three vertices in \( P \) for which the values of \( f \) are less than four (otherwise \( \sum_{v \in V(K_{5,6})} f(v) > 18 \)). If the number of vertices \( v \) in \( P \) for which \( f(v) \leq 3 \) is at least four, then the assumptions of Lemma 22 with \( k = 3 \) are
satisfied. Otherwise the assumptions of Lemma 22 with \( k = 1 \) hold. It contradicts the \((f, D_1)\)-choosability of \( K_{5,6} \) and proves \( \chi_{ac}^{D_1}(K_{5,6}) = 19 \).

The method of proving \( \chi_{ac}^{D_1}(K_{5,7}) = 21 \) and \( \chi_{ac}^{D_1}(K_{5,8}) = 23 \) is similar to this one, which we used to state \( \chi_{ac}^{D_1}(K_{5,6}) = 19 \). First we apply Theorem 21 to decide \( \chi_{ac}^{D_1}(K_{5,7}) \leq 21 \) and \( \chi_{ac}^{D_1}(K_{5,8}) \leq 23 \), respectively. Next, by the reasoning presented before we conclude that if there exists a size function \( f \) satisfying \( \sum_{v \in V(K_{5,7})} f(v) = 20 \), respectively \( \sum_{v \in V(K_{5,8})} f(v) = 22 \) such that \( K_{5,7} \), respectively \( K_{5,8} \) is \((f, D_1)\)-choosable and \((P, Q)\) is the bipartition of the corresponding complete bipartite graph with \(|P| = 5\), then \( f(v) = 1 \) for each \( v \in Q \). Next we can apply Lemma 22 with either \( k = 1 \) or \( k \in \{2, 3\} \) to confirm the \((f, D_1)\)-nonchoosability of \( K_{5,7} \) and apply the same lemma with either \( k = 1 \) or \( k \in \{2, 3, 4\} \) to confirm the \((f, D_1)\)-nonchoosability of \( K_{5,8} \). It implies that \( \chi_{ac}^{D_1}(K_{5,7}) = 21 \) and \( \chi_{ac}^{D_1}(K_{5,8}) = 23 \), respectively. Note that the reasoning associated with the conclusion on the \( D_1 \)-sum-choice-number of \( K_{5,7} \) has to be finished before we start with the corresponding reasoning connected with \( K_{5,8} \) because we need the equality \( \chi_{ac}^{D_1}(K_{5,7}) = 21 \) in order to use the trick of assuming that in \( K_{5,8} \) for all \( v \in Q \) it holds \( f(v) = 1 \).

Finally, by the application of Corollary 6, step by step, for \( j \in \{9, 10, 11, 12\} \) with \( G_1 = K_{5,j} \) and \( G_2 = K_{5,j-1} \) and by the application of Theorem 21 in all these cases we have \( \chi_{ac}^{D_1}(K_{5,j}) = j + 15 \). Additionally, Theorem 23 implies \( \chi_{ac}^{D_1}(K_{5,j}) = j + 15 \) for \( j \geq 13 \), which finishes the proof.

At this stage of the discussion we formulate the supposition.

**Conjecture 1.** If \( p, q \in \mathbb{N} \), \( q \geq p \) and \( q \geq 2(p-1) \), then

\[
\chi_{ac}^{D_1}(K_{p,q}) = \left( \frac{p + 1}{2} \right) + q.
\]

Obviously, it is true for \( p \in \{1, 2, 3, 4, 5\} \) (Corollary 11, Theorems 26, 27) and partially confirmed in other cases by Theorem 23.

Now we are in a position to prove a general fact on all bipartite graphs that will be preceded by the supporting observation.

**Lemma 28.** Let \( n \in \mathbb{N} \), \( n \geq 4 \). If \( f(x) = \left( \frac{x+1}{2} \right) + n - x - \left( x - \left( \frac{n-x}{2} \right) \right) \) and \( x \) is a positive integer satisfying \( 1 \leq x \leq \left\lfloor \frac{n}{2} \right\rfloor \), then

\[
f(x) \leq f\left( \left\lfloor \frac{n}{2} \right\rfloor \right).
\]

**Proof.** Let us assume that \( 1 \leq x \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Below we analyse the growth of the
function \( f \).
\[
f(x + 1) - f(x) = x + \frac{1}{2} \left( x - \left\lfloor \frac{n - x}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor - 1 \right) - \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor + 1 \right).
\]

Hence if \( n - x \) is an even number, then
\[
f(x + 1) - f(x) = x + 1 - \frac{1}{2} \left( x - \left\lfloor \frac{n - x}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor - 1 \right) - \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor + 1 \right) = n - 2x - 1 \geq n - 2 \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 0.
\]

If \( n - x \) is an odd number, then
\[
f(x + 1) - f(x) = x + 1 - \frac{1}{2} \left( x - \left\lfloor \frac{n - x}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor - 1 \right) - \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor \right) \left( x - \left\lfloor \frac{n - x - 1}{2} \right\rfloor + 1 \right) = \frac{1}{2} \left( n - x - 1 \right) \geq \frac{1}{2} \left( n - \left\lfloor \frac{n}{2} \right\rfloor \right) \geq 0.
\]

It follows that the function \( f(x) \) is nondecreasing for integers in \( \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \), which implies the assertion.

**Theorem 29.** If \( n \in \mathbb{N}, n \geq 2 \) and \( G \) is an \( n \)-vertex bipartite graph, then
\[
\chi_{D_1}^{sc}(G) \leq \frac{3}{32} n^2 + an + b,
\]
where
\[
(a, b) = \begin{cases} 
\left( \frac{28}{32}, 0 \right), & \text{if } n \equiv 0 \pmod{4} \\
\left( \frac{22}{32}, \frac{7}{32} \right), & \text{if } n \equiv 1 \pmod{4} \\
\left( \frac{24}{32}, \frac{4}{32} \right), & \text{if } n \equiv 2 \pmod{4} \\
\left( \frac{26}{32}, \frac{9}{32} \right), & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** When \( 2 \leq n \leq 3 \), then each \( n \)-vertex bipartite graph is acyclic. Thus, by Corollary 11, \( \chi_{D_1}^{sc}(G) = n \).

Let us assume that \( n \geq 4 \). Corollary 7 and Theorem 21 imply
\[
\chi_{D_1}^{sc}(G) \leq \chi_{D_1}^{sc}(K_{x,n-x}) \leq \left( \frac{x + 1}{2} \right) + n - x - \left( x - \left\lfloor \frac{n-x}{2} \right\rfloor \right) = f(x),
\]
for some \( x \in \{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \). Lemma 28 implies \( f(x) \leq f\left( \left\lfloor \frac{n}{2} \right\rfloor \right) \).
We obtain the conclusion by the calculation of \( f\left( \left\lfloor \frac{n}{2} \right\rfloor \right) \). Note that the characteristic of the number \( n \) with respect to the divisibility by four gives us four different cases.

### 3.2. Complete graphs

The remaining part of the paper investigates \( \chi_{D_1}^{D_1}(K_n) \) for each \( n \in \mathbb{N} \). The proof of the supporting lemma imitates one of the proofs given in [12].

**Lemma 30.** Let \( n \in \mathbb{N} \) and let \( f : V(K_n) \to \mathbb{N} \). If \( K_n \) is \( (f, D_1) \)-choosable and for \( t_1 \leq \cdots \leq t_n \), the multisets \( \{t_1, \ldots, t_n\} \) and \( \{f(v) : v \in V(K_n)\} \) are identical, then \( t_i \geq \left\lfloor \frac{i}{2} \right\rfloor \) for each \( i \in \{1, \ldots, n\} \).

**Proof.** For a contradiction, let \( j \) be the least index satisfying \( t_j < \left\lfloor \frac{j}{2} \right\rfloor \) and let \( L \) be a list assignment consisting of the sets \( \{1, \ldots, t_i\} \) for \( i \in \{1, \ldots, n\} \). In every \( (L, D_1) \)-colouring \( c \) of \( K_n \) we may use less than \( \frac{j+1}{2} \) colours for odd \( j \) and less than \( \frac{j}{2} \) colours for even \( j \) to colour the vertices which have assigned lists with sizes \( t_1, \ldots, t_j \). Hence \( c \) has an average number of monochromatic vertices, taken over the set \( \{v_1, \ldots, v_j\} \), equal to \( \frac{j}{2} \) for odd \( j \) and \( \frac{j}{2} \) for even \( j \), respectively. It means that there exist three vertices that have the same colour in \( c \). Because any three vertices of a complete graph induce a cycle it contradicts the assumed features of \( c \).

**Theorem 31.** If \( n \in \mathbb{N} \), then

\[
\chi_{D_1}^{D_1}(K_n) = \begin{cases} 
\frac{n(n+2)}{4}, & \text{if } n \text{ is even} \\
\frac{(n+1)^2}{4}, & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let \( v_1, \ldots, v_n \) be an arbitrary ordering of \( V(K_n) \) and let \( G_i = K_n[\{v_1, \ldots, v_i\}] \) for \( i \in \{1, \ldots, n\} \). Regardless of a choice of the ordering we have

\[
\sum_{i=1}^{n} \left( \left\lfloor \frac{\deg_G(v_i)}{2} \right\rfloor + 1 \right) = \begin{cases} 
2 \left( 1 + \cdots + \frac{n}{2} \right) = \frac{n(n+2)}{4}, & \text{if } n \text{ is even} \\
2 \left( 1 + \cdots + \frac{n-1}{2} \right) + \frac{n+1}{2} = \frac{(n+1)^2}{4}, & \text{if } n \text{ is odd}.
\end{cases}
\]

Because of \( \delta(D_1) = 2 \), Corollary 9 implies the upper bound on \( \chi_{ac}^{D_1}(K_n) \) given in the theorem.

By Lemma 30, for any mapping \( f : V(K_n) \to \mathbb{N} \) such that \( K_n \) is \( (f, D_1) \)-choosable, we have

\[
\sum_{v \in V(G)} f(v) \geq \frac{n(n+2)}{4} \quad \text{when } n \text{ is even and} \quad \sum_{v \in V(G)} f(v) \geq \frac{(n+1)^2}{4}
\]

when \( n \) is odd. It implies the lower bound on \( \chi_{ac}^{D_1}(K_n) \) and confirms the statement equality.
The *clique number* \( \omega(G) \) of a graph \( G \) is the largest number of vertices in its complete subgraph. As an immediate consequence of Theorem 31 we have the following fact connected with this notion.

**Corollary 32.** If \( G \) is an \( n \)-vertex graph with \( \omega(G) = \omega \), then

\[
\chi_{sc}^{D_1}(G) \geq \begin{cases} 
\frac{\omega(\omega - 2)}{4} + n, & \text{if } \omega \text{ is even} \\
\frac{(\omega - 1)^2}{4} + n, & \text{if } \omega \text{ is odd}.
\end{cases}
\]

**Proof.** By Corollary 6 we have \( \chi_{sc}^{D_1}(G) \geq \chi_{sc}^{D_1}(K_\omega) + n - \omega \). Theorem 31 implies the conclusion.

### 4. OPEN PROBLEMS

There are many interesting open questions connected with the subject of the paper. Below we present some of them.

1. For which \( n \)-vertex graphs \( G \) and (induced) hereditary classes of graphs \( P \) does there exist an ordering \( v_1, \ldots, v_n \) of \( V(G) \) such that

\[
\chi_{sc}^P(G) = n \sum_{i=1}^{n} \deg_H(v_i) + n,
\]

where \( H \) is the hypergraph of \( G \) with respect to \( P \) and \( H_i = H[\{v_1, \ldots, v_i\}] \)?

2. What are the exact values of \( \chi_{sc}^P(K_{p,q}) \) for \( p, q \in \mathbb{N} \) and for an arbitrary (induced) hereditary class of graphs \( P \) that is different from \( \mathcal{O} \) and different from \( D_1 \)?

3. What are the exact values of \( \chi_{sc}^{D_1}(K_{p,q}) \), when \( p, q \in \mathbb{N} \), \( q \geq p \geq 6 \) and \( q \leq \left(\frac{p+2}{2}\right) - 3 \)?

4. Is it true that if \( p, q \in \mathbb{N} \), \( q \geq p \) and \( q \geq 2(p-1) \), then

\[
\chi_{sc}^{D_1}(K_{p,q}) = \left(\frac{p+1}{2}\right) + q^2.
\]

5. What is the relationship between the \( D_1 \)-sum-choice-number of the Cartesian product of graphs and the \( D_1 \)-sum-choice-numbers of its factors, in particular the relationship between \( \chi_{sc}^{D_1}(P_n \Box P_m) \) and \( n, m \)?

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