HIGHER ORDER BELL POLYNOMIALS AND THE RELEVANT INTEGER SEQUENCES

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Dedicated to the Memory of Dr. Massimo Marchetti
an unforgettable dear friend

The recurrence relation for the coefficients of higher order Bell polynomials, i.e. of the Bell polynomials relevant to $n$th derivative of a multiple composite function, is proved. Therefore, starting from this recurrence relation and by using the computer algebra program Mathematica®, some tables for complete higher order Bell polynomials and the relevant numbers are derived.

1. INTRODUCTION

The Bell polynomials [3] are a mathematical tool for representing the $n$th derivative of a composite function. They are strictly related to partitions [1], [2], [21].

Several applications of the classical Bell polynomials have been considered in [5], [7], [9] (in connection with [22]), [13], [14].

Some generalized forms of Bell polynomials appeared in literature, see e.g. [11], [20]. Further generalizations can be found in [15], [16], and for the multidimensional case in [6], [19].

In particular, in [15], the higher order Bell polynomials and their main properties were introduced and recently, in [18], a recursion formula for the polynomial coefficients $A_{n,k}$ of the classical Bell polynomials was derived. This last result allows to compute the complete Bell polynomials $B_n$ and the relevant Bell numbers $b_n$, for every integer $n$.

In this article, after recalling this theory, and by using a more compact notation borrowed from [6], we prove the recurrence relation formula for the polynomial

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coefficients $A_{n,k}^{[r]}$ of the $r$th order Bell polynomials, generalizing in this way the result obtained in [18]. Therefore, by using this formula and by means of the computer algebra program Mathematica, we can obtain, for all integer $n$, tables of every order complete Bell polynomials $B_{n}^{[r]}$ and the relevant Bell numbers $b_{n}^{[r]}$. Here, as examples, we consider the cases $r = 2, 3, 4, 5$.

It is worth to note that the higher order Bell numbers appeared in literature as the McLaurin coefficients of a particular nested exponential function, while in our approach they assume a more general meaning. To our knowledge, tables of higher order Bell polynomials were never considered at all.

2. RECALLING THE BELL POLYNOMIALS

We recall that the Bell polynomials are a classical mathematical tool for representing the $n^{th}$ derivative of a composite function. In fact by considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$ defined in suitable intervals of the real axis and $n$ times differentiable with respect to the relevant independent variables and by using the following notations:

$\Phi_{h} := D_{t}^{h} \Phi(t), \quad f_{h} := D_{x}^{h} f(x)|_{x=g(t)}, \quad g_{h} := D_{t}^{h} g(t), \quad (1)$

and

$([f,g]_{n}) := (f_{1}, g_{1}; f_{2}, g_{2}; \ldots; f_{n}, g_{n}), \quad (2)$

they are defined as follows

$Y_{n}([f,g]_{n}) := \Phi_{n}. \quad (3)$

For example one has:

$Y_{1}([f,g]_{1}) = f_{1}g_{1},$
$Y_{2}([f,g]_{2}) = f_{1}g_{2} + f_{2}g_{1}^{2},$
$Y_{3}([f,g]_{3}) = f_{1}g_{3} + f_{2}(3g_{2}g_{1}) + f_{3}g_{1}^{3}.$

Further examples can be found in [21, p. 49].

Inductively, using the notation

$[g]_{n} := (g_{1}, g_{2}, \ldots, g_{n}),$

we can write:

$Y_{n}([f,g]_{n}) = \sum_{k=1}^{n} A_{n,k}([g]_{n}) f_{k}, \quad (4)$

where the coefficient $A_{n,k}$, for any $k = 1, \ldots, n$, is a polynomial in $g_{1}, g_{2}, \ldots, g_{n}$, homogeneous of degree $k$ and isobaric of weight $n$ (i.e. it is a linear combination of monomials $g_{1}^{k_{1}}g_{2}^{k_{2}}\cdots g_{n}^{k_{n}}$ whose weight is constantly given by $k_{1} + 2k_{2} + \ldots + nk_{n} = n$).

For them the following result holds true:
Proposition 1. The Bell polynomials satisfy the recurrence relation:

\[
\begin{cases}
Y_0 ([f, g]_0) := f_1 \\
Y_{n+1} ([f, g]_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k} ([f_1, g]_{n-k}) \ g_{k+1},
\end{cases}
\]

where

\([f_1, g]_{n-k} := (f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}).\]

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula \([10]\):

\[
\Phi_n = Y_n ([f, g]_n) = \sum_{\pi(n)} \frac{n!}{j_1! j_2! \ldots j_n!} \ f_{j_1} \frac{[g_1]^{j_1}}{1!} \ [g_2]^{j_2} \ [g_3]^{j_3} \ \cdots \ [g_n]^{j_n},
\]

where the sum runs over all partitions \(\pi(n)\) of the integer \(n\) (i.e. \(n = j_1 + 2j_2 + \cdots + nj_n\)), \(j_h\) denotes the number of parts of size \(h\) and \(j = j_1 + j_2 + \cdots + j_n\) denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in \([21]\). In \([23]\) the proof is based on the umbral calculus (see \([24]\) and the references therein).

The following result gives us a recursion formula for the coefficients \(A_{n,k}\) which appear in the Bell formula \((4)\) and are known as partial Bell polynomials. It was proved in \([18]\), but we will observe that it derives as a particular case of Theorem 7 proved here in Section 3.

Theorem 2. We have, \(\forall n:\)

\[
A_{n+1,1} = g_{n+1}, \quad A_{n+1,n+1} = g_1^{n+1}.
\]

Furthermore, \(\forall k = 1, 2, \ldots, n - 1\), the \(A_{n,k}\) coefficients can be computed by the recurrence relation

\[
A_{n+1,k+1} ([g]_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k} ([g]_{n-h}) \ g_{n+1}.
\]

The complete Bell polynomials, considered in literature, are defined by

\[
B_n ([g]_n) = Y_n (1, g_1; 1, g_2; \ldots; 1, g_n) = \sum_{k=1}^{n} A_{n,k} ([g]_n),
\]

and the Bell numbers by

\[
b_n = Y_n (1, 1; 1, 1; \ldots; 1, 1) = \sum_{k=1}^{n} A_{n,k} (1, 1, \ldots, 1).
\]
3. BELL POLYNOMIALS OF ORDER \( r \)

In [6] the following extension of the classical Bell polynomials was achieved. Consider \( \Phi(t) := f(\varphi^{(1)}(\varphi^{(2)}(\cdots(\varphi^{(r)}(t))))), \) i.e. the composition of functions \( x^{(r)} = \varphi^{(r)}(t), \ldots, x^{(2)} = \varphi^{(2)}(x^{(1)}), x^{(1)} = \varphi^{(1)}(x^{(2)}), y = f(x^{(1)}) \) defined in suitable intervals of the real axis, and suppose that the functions \( \varphi^{(r)}, \ldots, \varphi^{(2)}, \varphi^{(1)} \) are \( n \) times differentiable with respect to the relevant independent variables so that, by using the chain rule, \( \Phi(t) \) can be differentiated \( n \) times with respect to \( t \).

We use the following notations:

\[
\Phi_h := D^h_t \Phi(t), \\
f_h := D^h_t f(x^{(1)} = \varphi^{(1)}(\cdots(\varphi^{(r)}(t)))), \\
(11) \varphi^{(1)}_h := D^h_{x(2)} \varphi^{(1)}(x^{(2)} = \varphi^{(2)}(\cdots(\varphi^{(r)}(t)))), \\
\ldots \ldots \ldots, \varphi^{(r)}_h := D^h_{x(1)} \varphi^{(r)}(t),
\]

and

\[
\left( f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right)_n := (f_1, \varphi_1^{(1)}, \ldots, \varphi^{(r)}_1; \ldots; f_n, \varphi^{(1)}_n, \ldots, \varphi^{(r)}_n).
\]

Then the \( n^{th} \) derivative of the function \( \Phi \) allows us to define the (one-dimensional) Bell polynomials of order \( r \), \( Y^{[r]}_n \), as follows:

\[
(12) \quad Y^{[r]}_n \left( [f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n \right) := \Phi_n.
\]

For \( r = 1 \) we obtain the ordinary Bell polynomials \( Y^{[1]}_n \left( [f, \varphi^{(1)}]_n \right) = Y_n \left( [f, \varphi^{(1)}]_n \right) \). Note that we are considering here the one-dimensional case, while in [6] even the multi-dimensional Bell polynomials were introduced.

The first polynomials have the following explicit expressions:

\[
(13) \quad Y^{[r]}_1 \left( [f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_1 \right) = f_1 \varphi^{(1)}_1 \cdots \varphi^{(r)}_1,
\]

\[
Y^{[r]}_2 \left( [f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_2 \right) = f_2 \left( \varphi^{(1)}_1 \cdots \varphi^{(r)}_1 \right)^2 + f_1 \varphi^{(1)}_2 \left( \varphi^{(2)}_1 \cdots \varphi^{(r)}_1 \right)^2
\]

\[
+ f_1 \varphi^{(1)}_1 \varphi^{(2)}_2 \left( \varphi^{(3)}_1 \cdots \varphi^{(r)}_1 \right)^2 + f_1 \varphi^{(1)}_1 \varphi^{(2)}_1 \varphi^{(3)}_2 \cdots \varphi^{(r-1)}_1 \varphi^{(r)}_2.
\]

In general, we have

\[
(14) \quad Y^{[r]}_n ([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n) = \sum_{k=1}^{n} A^{[r]}_{n,k} ([\varphi^{(1)}, \ldots, \varphi^{(r)}]_n) f_k.
\]

Some useful properties, proved in [15], satisfied by the polynomials \( Y^{[r]}_n \) are the following:
Theorem 3. For every integer $n$, the polynomials $Y_n^r$ are expressed in terms of the Bell polynomials of lower order, by means of the following equation:

\[
Y_n^r \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right) = Y_n \left( \left[ f, Y_n^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right) \right]_n \right),
\]

where

\[
\left( \left[ f, Y_n^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right) \right]_n \right) := \left( f_1, Y_1^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_1 \right) \right); \ldots; f_n, Y_n^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right).
\]

Theorem 4. The following recurrence relation for the Bell polynomials $Y_n^r$ holds true:

\[
\begin{cases}
Y_0^r \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_0 \right) = f_1 \\
Y_n^r \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) = \sum_{k=0}^{n} \binom{n}{k} \times Y_{n-k} \left( \left[ f_1, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-k} \right) Y_{k+1}^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{k+1} \right),
\end{cases}
\]

where

\[
\left( \left[ f_1, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-k} \right) := \left( f_2, \varphi^{(1)}_1, \ldots, \varphi^{(r)}_1; \ldots; f_n-k+1, \varphi^{(1)}_{n-k}, \ldots, \varphi^{(r)}_{n-k} \right).
\]

Theorem 5. The generalized Faà di Bruno formula holds true:

\[
\begin{align*}
Y_n^r \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right) &= \sum_{\pi(n)} \frac{n!}{j_1!j_2! \ldots j_n!} \cdot f_1 \left[ Y_1^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_1 \right) \right]^{j_1} \\
&\quad \times \frac{Y_2^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_2 \right)}{2!}^{j_2} \ldots \left[ Y_n^{r-1} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_n \right) \right]^{j_n}.
\end{align*}
\]

By putting, for every integer $s$ ($1 \leq s \leq r-1$),

\[
\varphi^{(s+1)}(\varphi^{(s+2)}(\ldots(\varphi^{(r)}(t))\ldots)) =: g(t), \quad f(\varphi^{(1)}(\ldots(\varphi^{(s)}(x))\ldots)) =: f(x)
\]

where $x = g(t)$, the composite function $\Phi(t) := f(\varphi^{(1)}(\cdots(\varphi^{(r)}(t))\cdots))$, can be written as follows

\[
\Phi(t) = f(g(t)).
\]

Therefore the following result holds true:
Theorem 6. For every integer \( n \), the polynomials \( Y_n^{[r]} \) are expressed in terms of the Bell polynomials of lower order, by means of the following equation:

\[
Y_n^{[r]} ([f, \varphi^{(1)}, \ldots, \varphi^{(r)}])_n = Y_n \left( Y^{[s]} ([f, \varphi^{(1)}, \ldots, \varphi^{(s)}]), Y^{[r-s-1]} ([\varphi^{(s+1)}, \ldots, \varphi^{(r)}])_n \right).
\]

(18)

The complete Bell polynomials of order \( r \), \( B_n^{[r]} \), are defined by the equation:

\[
B_n^{[r]} ([\varphi^{(1)}, \ldots, \varphi^{(r)}])_n = Y_n^{[r]} (1, \varphi_1^{(1)}, \ldots, \varphi_1^{(r)}; \ldots; 1, \varphi_n^{(1)}, \ldots, \varphi_n^{(r)})
\]

\[
= \sum_{k=1}^{n} A_{n,k}^{[r]} \left( [\varphi^{(1)}, \ldots, \varphi^{(r)}]_n \right),
\]

and the \( r \)th order Bell numbers by

\[
b_n^{[r]} = Y_n^{[r]} (1, 1, 1; \ldots; 1, 1, 1) = \sum_{k=1}^{n} A_{n,k}^{[r]} (1; \ldots; 1, 1).
\]

Now, in order to derive tables for complete higher order Bell polynomials \( B_n^{[r]} \) and the relevant higher order Bell numbers \( b_n^{[r]} \), we generalize the result given in Theorem 2, by means of the following theorem

Theorem 7. We have, \( \forall n \)

\[
A_{n+1,1}^{[r]} = Y_{n+1}^{[r-1]} \left( [\varphi^{(1)}, \ldots, \varphi^{(r)}]_{n+1} \right),
\]

(19)

\[
A_{n+1,n+1}^{[r]} = \left( Y_{n+1}^{[r-1]} ([\varphi^{(1)}, \ldots, \varphi^{(r)}])_1 \right)^{n+1} = \left( \varphi_1^{(1)} \ldots \varphi_1^{(r)} \right)^{n+1}.
\]

Furthermore, \( \forall k = 1, 2, \ldots, n-1 \), the \( r \)-th order partial Bell polynomials \( A_{n,k}^{[r]} \) satisfy the recursion:

\[
A_{n+1,k+1}^{[r]} \left( [\varphi^{(1)}, \ldots, \varphi^{(r)}]_{n+1} \right) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}^{[r]} \left( [\varphi^{(1)}, \ldots, \varphi^{(r)}]_{n-h} \right)
\]

\[
\times Y_{h+1}^{[r-1]} \left( [\varphi^{(1)}, \ldots, \varphi^{(r)}]_{h+1} \right).
\]

(20)
Proof. According to equations (14) and (4), using Theorem 3, we can write

\[ Y_{n+1}^{[r]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) = \sum_{k=1}^{n+1} A_{n+1,k}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_k \]

\[ = Y_{n+1} \left( f_1, Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_1 \right); \ldots; f_{n+1}, Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) \right) \]

\[ = \sum_{k=1}^{n+1} A_{n+1,k} \left( Y_{1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_1 \right); \ldots; Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) \right) f_k, \]

so that we find the following relations between the classical polynomial coefficients and the \( r \)-th order ones:

\[ A_{n+1,k}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) \]

\[ = A_{n+1,k} \left( Y_{1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_1 \right); \ldots; Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) \right). \]

Equations (19) can be obtained from relations (21), for \( k = 1 \) and \( k = n + 1 \), as a direct consequence of the definition of the ordinary coefficient \( A_{n+1,k} \) given in (4). In order to prove equation (20), note that, taking into account the first relation in (19), we can write the equation (14) in the form

\[ Y_{n+1}^{[r]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) = \sum_{k=0}^{n} A_{n+1,k+1}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_{k+1} \]

\[ = A_{n+1,1}^{[r]} f_1 + \sum_{k=1}^{n} A_{n+1,k+1}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_{k+1} \]

\[ = Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_1 + \sum_{k=1}^{n} A_{n+1,k+1}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_{k+1}. \]

Furthermore, recalling equation (16)_1, the equation (16)_2 becomes
\[ \begin{align*}
Y_{r+1}^{[r]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) &= \sum_{h=0}^{n} \binom{n}{h} Y_{n-h}^{[r]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-h} \right) Y_{h+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{h+1} \right) \\
&= f_1 Y_{n+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) \\
&+ \sum_{h=0}^{n-1} \binom{n}{h} Y_{n-h}^{[r]} \left( \left[ f, \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-h} \right) Y_{h+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{h+1} \right) 
\end{align*} \]

so that, neglecting the first term in both the above sums, we find:

\[ \begin{align*}
\sum_{k=1}^{n} A_{n+1,k+1}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n+1} \right) f_{k+1} &= \sum_{h=0}^{n-1} \binom{n}{h} \left( \sum_{\ell=1}^{n-h} A_{n-h,\ell}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-h} \right) f_{\ell+1} \right) Y_{h+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{h+1} \right) \\
&= \sum_{\ell=1}^{n} \left( \sum_{h=0}^{n-\ell} \binom{n}{h} A_{n-h,\ell}^{[r]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{n-h} \right) Y_{h+1}^{[r-1]} \left( \left[ \varphi^{(1)}, \ldots, \varphi^{(r)} \right]_{h+1} \right) \right) f_{\ell+1}. 
\end{align*} \]

Therefore, changing \( \ell \) into \( k \) in the last formula and equating the coefficients of \( f_{k+1} \), the equation (20) follows.

### 4. TABLES OF COMPLETE HIGHER ORDER BELL POLYNOMIALS, FOR \( r = 2, 3, 4, 5 \)

By using the recurrence relation (19)-(20) and by means of the computer algebra program Mathematica\textsuperscript{®} we can construct the complete Bell polynomials of every order. We will limit ourselves to present here the very first of them. Putting, for shortness: \( x = \varphi^{(1)} \), \( y = \varphi^{(2)} \), \( z = \varphi^{(3)} \), \( u = \varphi^{(4)} \), \( v = \varphi^{(5)} \), and denoting by indices the order of derivatives, we have found:
Second order Bell polynomials:

\[ B_2^3(x, y) = x_1y_1; \]
\[ B_2^3(x, y) = x_1^2y_1^2 + x_2y_1^2 + x_1y_2; \]
\[ B_3^3(x, y, z) = x_1^3y_1^3 + 3x_1x_2y_1^3 + x_1y_1^3 + 3x_1^2y_1y_2 + 3x_2y_1y_2 + x_1y_3; \]
\[ B_4^4(x, y, z) = x_1^4y_1^4 + 6x_1^3x_2y_1^4 + 3x_1^2x_2^2y_1^4 + 4x_1x_2x_3y_1^4 + 4x_4y_1^4 + 6x_1^3y_1^2y_2 + 18x_1^2x_2y_1^2y_2 + 6x_1y_1^2y_2 + 3x_2y_1^2y_2 + 3x_2y_2^2 + 4x_1^2y_1^2y_3 + 4x_2y_1^2y_3 + x_1y_4; \]
\[ B_5^5(x, y, z) = x_1^5y_1^5 + 10x_1^4x_2y_1^5 + 15x_1^3x_2^2y_1^5 + 10x_1^2x_2^3y_1^5 + 5x_1^3y_1y_2^5 \]
\(+ 5x_1^2x_2y_1y_2^5 + 60x_1^2x_2^2y_1y_2^5 + 30x_1^2y_1^2y_2^2y_4 + 20x_1y_1y_2y_3y_4 + 10x_1^2y_1^2y_3^2 + 10x_1y_1^2y_2y_3 + 10x_1y_1y_2y_4 + 10x_1^2y_1^2y_2 + 10x_2y_1y_2^2y_3 + 5x_1^2y_1^2y_4 + 5x_2y_1y_3y_4 + x_1y_5. \]

Third order Bell polynomials:

\[ B_3^3(x, y, z) = x_1^3y_1^3 + 3x_1^2y_1y_2 + 3x_1y_1^2 + 3x_1^3 + 3x_1^2y_2 + 3x_1y_1y_3 + 3x_1^2y_3 + 3x_1y_2^2 + 3x_1y_1^3; \]
\[ B_4^4(x, y, z) = x_1^4y_1^4 + 3x_1^3y_1y_2 + 4x_1^2y_1y_2 + 4x_1y_1^2 + 4x_1y_2^2 + 4x_1y_1^3 + 3x_1^2y_1 + 3x_1y_1y_3 + 3x_1^2y_2 + 3x_1y_2^2 + 3x_1y_3^2; \]
\[ B_5^5(x, y, z) = x_1^5y_1^5 + 3x_1^4y_1y_2 + 6x_1^3y_1y_2 + 6x_1^2y_1y_2 + 6x_1y_1y_2 + 6y_1^5 + 30x_1^3y_1y_3 + 30x_1^2y_1y_3 + 30y_1^3y_2^2 + 30y_1^2y_2^2 + 30y_1y_2^2 + 3y_2^5 + 15x_1^4y_2 + 45x_1^3y_2 + 15x_1^2y_2 + 15x_1y_2 + 10x_2y_2 + 10x_1^3y_3 + 10x_1^2y_3 + 10x_1y_3 + 10y_3^2 + 10x_1^2y_4 + 10x_2y_4 + 10x_1y_4 + 5x_1y_5; \]

Fourth order Bell polynomials:

\[ B_4^4(x, y, z, u) = u_1^4z_1y_1^4 + 3u_1^3z_2y_1^4 + 4u_1^2u_2z_2y_1^4 + 4u_1u_3z_2y_1^4 + 6u_2^2z_2y_1^4; \]
\[ B_5^5(x, y, z, u) = u_1^5z_1y_1^5 + 4u_1^4z_2y_1^5 + 4u_1^3u_2z_2y_1^5 + 4u_1^2u_3z_2y_1^5 + 4u_1u_4z_2y_1^5 + 4u_4^2z_2y_1^5 + 10u_1^3y_1^2 + 10u_1^2u_2y_1^2 + 10u_1u_3y_1^2 + 10u_4y_1^2 + 10y_1^3 + 10u_1^2z_2 + 10u_2^2z_2 + 10u_1u_2z_2 + 10u_1z_2^2 + 10u_2z_2^2 + 10u_4z_2^2 + 10z_2^3; \]
Fifth order Bell polynomials:

\[ B_5^5 \left( (x, y, z, u, v) \right) = v_1 u_1 z_1 y_1 x_1; \]

\[ B_2^2 \left( (x, y, z, u, v) \right) = v_2 u_1 z_1 y_1 x_1 + v_1^2 u_2 z_1 y_1 x_1 + v_1^2 u_1 z_2 y_1 x_1 + v_1^2 u_1 z_1 y_2 x_1 + v_1^2 u_1 z_1 y_1 x_2; \]

\[ B_3^3 \left( (x, y, z, u, v) \right) = v_3 u_1 z_1 y_1 x_1 + 3 v_1 v_2 u_2 z_1 y_1 x_1 + v_1^3 u_3 z_1 y_1 x_1 + 3 v_1 v_2 u_2 z_1 y_1 x_1 + v_1^3 u_3 z_1 y_2 x_1 + v_1^3 u_3 z_2 y_1 x_1 + 3 v_1 v_2 u_2 z_1 y_2 x_1 + v_1^3 u_3 z_2 y_2 x_1 + v_1^3 u_3 z_3 y_1 x_1 + v_1^3 u_3 z_3 y_2 x_1 + v_1^3 u_3 z_3 y_3 x_1 \]

\[ B_4^4 \left( (x, y, z, u, v) \right) = v_4 u_1 z_1 y_1 x_1 + 3 v_1 v_3 u_2 z_1 y_1 x_1 + 4 v_1 v_3 u_2 z_2 y_1 x_1 + 4 v_1 v_3 u_3 z_1 y_1 x_1 + 4 v_1 v_3 u_3 z_2 y_1 x_1 + 4 v_1 v_3 u_3 z_3 y_1 x_1 \]

\[ + 18 v_1^2 v_2 u_1 z_1 y_1 x_1 + 3 v_1^2 u_2 z_1 y_1 x_1 + 4 v_1^2 u_3 z_1 y_2 x_1 + 4 v_1^2 u_3 z_3 y_1 x_1 + 4 v_1^2 u_3 z_3 y_2 x_1 + 4 v_1^2 u_3 z_3 y_3 x_1 \]

5. HIGHER ORDER BELL NUMBERS, FOR \( r = 2, 3, 4, 5 \)

It is worth to note that the sequences of higher order Bell numbers which will be presented here appear in the Encyclopedia of Integer Sequences [25] under the # A144150, arising from a problem of Combinatorial Analysis and even as the McLaurin coefficients of the functions \( [4], [12] \)

\[ \exp(\exp(\exp(x) - 1) - 1), \]
\[ \exp(\exp(\exp(x) - 1) - 1), \]
\[ \exp(\exp(\exp(x) - 1) - 1) - 1), \]
\[ \exp(\exp(\exp(\exp(x) - 1) - 1) - 1) - 1), \]

for the cases \( r = 2, r = 3, r = 4, r = 5, \) respectively, and so on for the subsequent values of \( r. \) Whereas in our approach they assume a more general meaning, as they are independent of the functions \( f, \varphi(1), \ldots, \varphi(r) \).
According to the above reference we have found, using the recurrence relation (19)-(20) and by means of the computer algebra program Mathematica®, the following sequences for the higher order Bell numbers $b_{[2]}^n$, $b_{[3]}^n$, $b_{[4]}^n$, $b_{[5]}^n$, $(n = 1, 2, \ldots, 21)$:

<table>
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<th>$n$</th>
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<th>$b_{[4]}^n$</th>
<th>$b_{[5]}^n$</th>
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REFERENCES


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